#1

確率空間 (Ω, B, P) ①

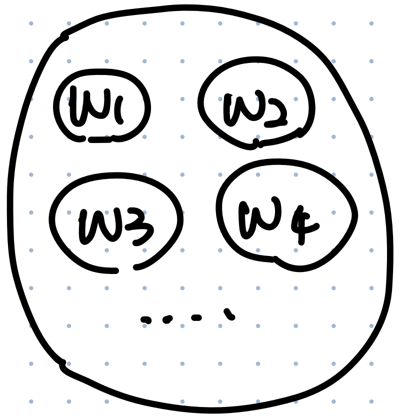
Trial.

↓ ← 標本点

ω_r

↓

$\{\omega_1 \cdots \omega_n\} = \Omega$.



↓

標本空間 Ω

⇒ 事象.

ϕ. Ω.

$A_1 = (\omega_1) = \{\omega_1\}$

$A_2 = (\omega_1, \omega_2) = \{\omega_1, \omega_2\}$

$A_3 = (\omega_1\ \omega_3) = \{\omega_1, \omega_3\}$

⋮

$\Rightarrow \{\phi, \Omega\}$

$\{\phi, \Omega, A_1, A_2 \cdots\}$

....

↓

σ-algebra: B.

可測集合族.

⇓

$P(\Omega) = 1$. $P(A_i) \geq 0$

$P(\bigcup_i A_i) = \sum_i P(A_i)$

↓

確率.

$P(\omega_i)$ $\exists \omega_i \notin \mathbb{R}$

↓ $X(\omega_i) = x_i$ X: 確率変数.

$P(x_i)$ $\forall x_i \in \mathbb{R}$

↓

(論)

関連 ① Xの確率 → PMF. CDF.

② Xの変動の記述 → 期待. 函数. $\xrightarrow{\text{一般化}}$ moment. (MGF)

Thm. 1.4.

① Taylor expansion of e^x. $\exp\{x\}$. (supp. 1.6)

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad (*)$$

$$e^{tX} = 1 + (tX) + \cdots = \sum_{k=0}^{\infty} \frac{(tX)^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k X^k}{k!}$$

$\forall t \in \mathbb{R}$

確率変数 (Ω.B.P)

② $$M_X(t) = E[e^{tX}] = E\left[\sum_{k=0}^{\infty} \frac{t^k X^k}{k!}\right] = \sum_{k=0}^{\infty} \frac{t^k}{k!} E[X^k]$$

(*) $f(x) = e^x \quad f(a) = e^a$

$f'(x) = e^x \quad f'(a) = e^a$

$f''(x) = e^x \quad f''(a) = e^a$

$\vdots \qquad \vdots$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3}(x-a)^3 + \cdots$$

$$e^x = e^a + e^a(x-a) + \frac{e^a}{2}(x-a)^2 + \cdots$$

$a=0$

$$e^x = e^0 + e^0(x-0) + \frac{e^0}{2!}(x-0)^2 + \frac{e^0}{3!}(x-0)^3 + \cdots$$

$$= 1 + (x) + \frac{1}{2!}(x^2) + \frac{1}{3!}(x^3) + \cdots$$

③ $M_X(t) = \frac{t^0}{0!}E[X^0] + \frac{t}{1!}E[X] + \frac{t^2}{2!}E[X^2] + \cdots$ $\qquad 0! = 1$

$$= 1 + tE[X] + \frac{t^2}{2!}E[X^2] + \frac{t^3}{3!}E[X^3] + \cdots$$

$$\frac{dM_X(t)}{dt} = E[X] + \frac{d}{dt}\{\cdots + \cdots\} \Rightarrow \left.\frac{dM_X(t)}{dt}\right|_{t=0} = E[X] + 0$$

$$\frac{d^2M_X(t)}{dt^2} = 0 + \left(\frac{2t}{2!}E[X^2]\right)' + \frac{d^2}{dt^2}\{\cdots\} \xrightarrow{t=0} = E[X^2]$$

$$\left(\frac{2t}{2!}E[X^2]\right)' \to \frac{2}{2!}E[X^2]$$

$$\frac{d^3M_X(t)}{dt^3} = 0 + 0 + \frac{3\times 2t}{3!}E[X^3] + \frac{d^3}{dt^3}\{\cdots\} \xrightarrow{t=0} = E[X^3]$$

$$\left(\frac{3t^2}{3!}\right)'E[X^3] \to \frac{3\times 2t}{3!}E[X^3]$$

$\cdots\cdots$

---

③' (連).

Proof.:

$$\frac{d}{dt}M_X(t) = \frac{d}{dt}\int e^{tx}f_X(x)\,dx$$

(連 & 微分可) Leibnitz's Rule.

$$= \int \frac{d}{dt}(e^{tx})\, f_X(x)\,dx$$

$$= \int x e^{tx} f_X(x)\,dx = E[Xe^{tX}] \overset{t=0}{=} E[X]$$

$$\frac{d^2}{dt^2}M_X(t) = \frac{d}{dt}\int x e^{tx} f_X(x)\,dx$$

$$= \int x \cdot \frac{de^{tx}}{dt} f_X(x)\,dx$$

$$= \int x^2 e^{tx} f_X(x)\,dx$$

$$= E[X^2 e^{tX}] \overset{t=0}{=} E[X^2]$$

Thm 1.6

$$Y = X_1 + X_2 + \cdots + X_n$$

$$M_Y(t) = E[e^{tY}]$$

$$= E[e^{t(X_1+X_2+\cdots+X_n)}]$$

$$= E[e^{tX_1} e^{tX_2} \cdots e^{tX_n}] \quad \text{独立}$$

$$= E[e^{tX_1}]E[e^{tX_2}] \cdots E[e^{tX_n}]$$

$$= M_{X_1}(t) \cdots M_{X_n}(t)$$

$$E[X] = \sum_{k=1}^{n} k \cdot \frac{1}{n} = \frac{1}{n} + \frac{2}{n} + \frac{3}{n} + \cdots + \frac{n}{n} = \frac{1}{n}(1+2+\cdots+n)$$

$$= \frac{1}{n} \cdot \frac{n(1+n)}{2} = \frac{1+n}{2}$$

$$E[X^2] = \sum_{k=1}^{n} k^2 \cdot \frac{1}{n} = \frac{1}{n} + \frac{2^2}{n} + \cdots + \frac{n^2}{n} = \frac{1}{n}(1^2+2^2+\cdots+n^2)$$

$$= \frac{1}{n} \cdot \frac{n(n+1)(2n+1)}{6}$$

$$Var[X] = E(X^2) - \{E[X]\}^2 = \frac{(n+1)(2n-1)}{6} - \left(\frac{n+1}{2}\right)^2 = \frac{n^2-1}{12}$$

Def. 1.12. MGF of discrete uniform distribution.

$$P(X=k) = \frac{1}{n}, \quad k=1,2,\ldots,n$$

$$\uparrow x_k \quad \uparrow p_k$$

$$M_X(t) = E[e^{tX}]$$

$$= \sum_{k=1}^{n} e^{tx_k} \cdot p_k$$

$$e^{ab} = (e^a)^b$$

$$= \sum_{k=1}^{N} e^{tk} \frac{1}{n}$$

$$= \sum_{k=1}^{N} \{e^t\}^k \frac{1}{n} \rightarrow e^t + e^t e^t + e^t e^t e^t + \cdots \times \frac{1}{n}$$

$$= \frac{1}{n} \cdot e^t \{1 + e^t + e^t e^t + \cdots + (e^t)^{n-1}\}$$

$$= \frac{e^t}{n} \sum_{k=0}^{n-1} \{e^t\}^k$$

$$\left(r = e^t \quad = \frac{r}{N} \sum_{k=0}^{n-1} r^k \rightarrow \text{等比数列和} \sum_{k=0}^{n-1} r^k = \frac{r^n - 1}{r - 1} \right)$$

Sum of geometric series

$$= \frac{e^t}{n} \left(\frac{e^{tn} - 1}{e^t - 1} \right)$$

Def. 1.13. MGF of Binomial distribution.

$$M_X(t) = E[e^{tX}] = \sum_{k=0}^{n} e^{tk}\binom{n}{k}p^k(1-p)^{n-k}$$

$$= \sum_{k=0}^{n}\binom{n}{k} e^{tk}p^k(1-p)^{n-k}$$

$$= \sum_{k=0}^{n}\binom{n}{k}(\underbrace{e^t p}_{y})^k(\underbrace{1-p}_{x})^{n-k}$$

$$= (\underbrace{e^t p}_{y} + \underbrace{1-p}_{x})^n$$

→ 二項定理 Binomial Theorem → $(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + y^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k$

① $$M_X'(t) = \frac{dM_X(t)}{dt} = n(e^t p + 1-p)^{n-1}\cdot p\cdot e^t$$

② $$M_X''(t) = \frac{d^2M_X(t)}{dt^2} = pn(n-1)(e^t p+1-p)^{n-2}\cdot p\cdot e^t\cdot e^t + pn(e^t p+1-p)^{n-1}\cdot e^t$$

(*) $f(g(x))' = f'(g(x))\cdot g'(x)$

$[f(x)\cdot g(x)]' = f'(x)g(x) + g'(x)f(x)$

$$E[X] = M_X'(0) = n(p+1-p)^{n-1}\cdot p = n\cdot p$$

$$E[X^2] = M_X''(0) = n(n-1)(p+1-p)^{n-2}\cdot p\cdot p + n(p+1-p)^{n-1}\cdot p$$
$$= n(n-1)\cdot p^2 + np$$

$$Var[X] = E[X^2] - \{E[X]\}^2 = n(n-1)p^2 + np - n^2p^2 = (n^2-n)p^2 + np - n^2p^2$$
$$= n^2p^2 - np^2 + np - n^2p^2 = np - np^2 = np(1-p)$$

Def. 1.14. MGF of Poisson distribution

$$M_X(t) = E[e^{tX}] = \sum_{k=0}^{\infty} e^{tk} \cdot \frac{e^{-\lambda}\lambda^k}{k!}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{e^{tk}\lambda^k}{k!}$$

$$= e^{-\lambda} \underbrace{\sum_{k=0}^{\infty} \frac{(\underbrace{e^t \cdot \lambda}_{x})^k}{k!}}$$

$$= e^{-\lambda} \underbrace{e^{\lambda e^t}}$$

$$= e^{(\lambda e^t - \lambda)}$$

$$= \exp\{\lambda(e^t - 1)\}$$

Taylor expansion. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$= \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$x = e^t \cdot \lambda$

$$e^{e^t \cdot \lambda} = \sum_{k=0}^{\infty} \frac{(e^t \cdot \lambda)^k}{k!}$$

① $M_X'(t) = \exp\{\lambda(e^t - 1)\} \cdot \lambda \cdot e^t$.

② $M_X''(t) = \exp\{\lambda(e^t - 1)\} \cdot \lambda \cdot e^t \cdot \lambda \cdot e^t$
$+ \exp\{\lambda(e^t - 1)\} \cdot \lambda \cdot e^t$.

$E[X] = M_X'(0) = \lambda$

$E[X^2] = M_X''(0) = \lambda^2 + \lambda$

$Var[X] = E(X^2) - \{E(X)\}^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$

Thm. 1.7. Poisson $\approx$ Binomial. 7

$X \sim \text{Poisson}(\lambda)$: $P(X=x) = \frac{\lambda}{x} P(X=x-1)$

$Y \sim \text{Bin}(n,p)$: $P(Y=y) = \frac{(n-y+1)}{y} \frac{p}{1-p} P(Y=y-1)$

$$\frac{\lambda}{x} P(X=x-1) = \frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!} \cdot \frac{\lambda}{x} = \frac{e^{-\lambda}\cdot\lambda^{x}}{x!} = P(X=x).$$

$$\frac{n-y+1}{y} \cdot \frac{p}{1-p} \binom{n}{y-1} p^{y-1}(1-p)^{n-y+1} = \underbrace{\frac{n-y+1}{y}\binom{n}{y-1}}_{(n-y+2)} \cdot p^{y}(1-p)^{n-y}$$

$$\frac{n(n-1)\cdots(n-(y-1)+1)(n-y+1)}{y\,(y-1)(y-2)\cdots 1} = \binom{n}{y}$$

Let. $\lambda = np$.

① $(y \neq 0)$

$$\frac{n-y+1}{y}\,\frac{p}{1-p} = \frac{np-yp+p}{y-yp} = \frac{\lambda-yp+p}{y-yp} \xrightarrow{p\to 0} \frac{\lambda}{y}.$$

$\Rightarrow P(Y=y) = \frac{\lambda}{y} P(Y=y-1)$

② $y=0$. $P(Y=0) = \binom{n}{0} p^0 (1-p)^n = (1-p)^n = (1-\frac{\lambda}{n})^n \xrightarrow{n\to\infty} e^{-\lambda} = \frac{e^{-\lambda}\cdot\lambda^0}{0!} = P(X=0)$

⊛ $\lim_{n\to\infty}(1-\frac{\lambda}{n})^n = e^{-\lambda}$

⊛ Lemma.

Define: $a_1, a_2, \cdots, a_n$ 数列

Suppose: $\lim_{n\to\infty} a_n = a$.

$\Rightarrow \lim_{n\to\infty}(1+\frac{a_n}{n})^n = e^a$

MGF: $M_Y(t) = \{(1-p)+pe^t\}^n \stackrel{p=\frac{\lambda}{n}}{=\!=} \{(1-\frac{\lambda}{n})+\frac{\lambda}{n}\cdot e^t\}^n$

$= \{1+\frac{1}{n}(e^t-1)\cdot\lambda\}^n$

$M_X(t) = \exp\{\lambda(e^t-1)\}$

Let $a_n = (e^t-1)\cdot\lambda = a$.

$\xrightarrow{n\to\infty} e^{\{(e^t-1)\cdot\lambda\}} = M_X(t)$ (Thm. 1.5)

Def. 1.15. MGF of geometric distribution.

$$M_X(t) = E[e^{tX}] = \sum_{k=1}^{\infty} e^{tk}(1-p)^{k-1}p. \quad (k = 1 \ldots \infty \quad e^{tk} : e^{t}, e^{2t}, e^{3t} \ldots)$$

$$= p\sum_{k=0}^{\infty} e^{t(k-1)}(1-p)^{k-1} \cdot e^{t}. \quad (k=0, \ldots \infty, \quad e^{t(k-1)} : e^{-t}, e^{t}, e^{2t}, e^{3t} \ldots)$$

$$= e^{t}p\sum_{k=0}^{\infty}\{e^{t}(1-p)\}^{k-1}$$

$$r = e^{t}(1-p) \quad = e^{t}p\sum_{k=0}^{\infty} r^{k-1} \longrightarrow \sum_{k=0}^{\infty} r^{k} = \frac{1}{1-r} \text{ if. } |r|<1$$

$$= e^{t}p \cdot \frac{1}{1-r}$$

$$= e^{t}p \cdot \frac{1}{1-e^{t}(1-p)}$$

(*) $r = |e^{t}(1-p)| < 1$: $p=1. \Rightarrow r=0$ satisfied.

$0<p<1$: $e^{t}(1-p)>0$

$e^{t}(1-p)<1$. then. satisfied.

$$M_X(t) = e^{t}p\{1-(1-p)e^{t}\}^{-1}$$

① $M_X'(t) = e^{t}p\{1-(1-p)e^{t}\}^{-1}$

$$+ e^{t}p(-1)\{1-(1-p)e^{t}\}^{-2}\{-(1-p)e^{t}\}$$

$(q = 1-p)$

$$= \frac{e^{t}p}{1-qe^{t}} + \frac{+e^{t}p(qe^{t})}{\{1-qe^{t}\}^{2}} = \frac{pe^{t}}{\{1-qe^{t}\}^{2}}$$

$$E[X] = p\{1-(1-p)\}^{-1}$$
$$-p\{1-(1-p)\}^{-2}$$
$$\{-(1-p)\}$$
$$= 1 - \frac{p(p-1)}{p^{2}}$$
$$\Big|_{t=0} = 1 - \frac{p-1}{p} = \frac{1}{p}$$

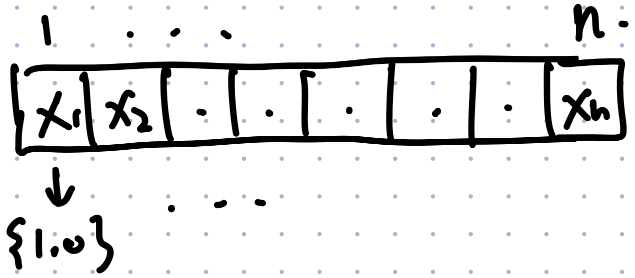
② $M_X''(t) = \dfrac{pe^{t}\cdot\{1-qe^{t}\}^{2} - pe^{t}\cdot 2\{1-qe^{t}\}\{-qe^{t}\}}{\{1-qe^{t}\}^{4}}$

$$= \frac{pe^{t} - pq^{2}e^{3t}}{(1-qe^{t})^{4}} \xrightarrow{t=0} E[X^{2}] = \frac{p-pq^{2}}{(1-q)^{4}} = \frac{p(1-q^{2})}{p(1-q)^{3}}$$

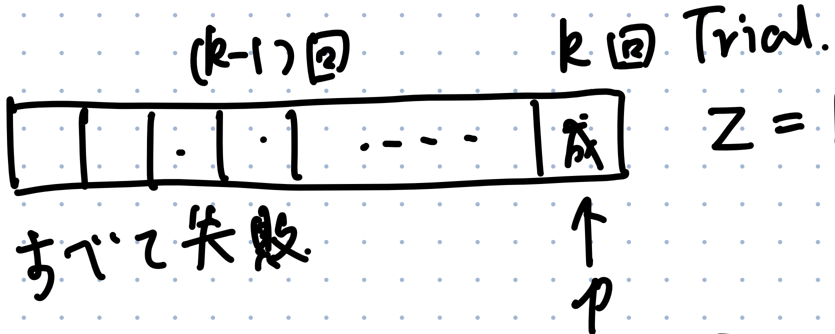
$$Var[X] = E[X^{2}] - \{E[X]\}^{2} = \frac{2-p}{p^{2}} - \frac{1}{p^{2}} = \frac{1+q}{(1-q)^{2}} = \frac{1+q}{p^{2}}$$

Bernoulli Trial に関する確率分布

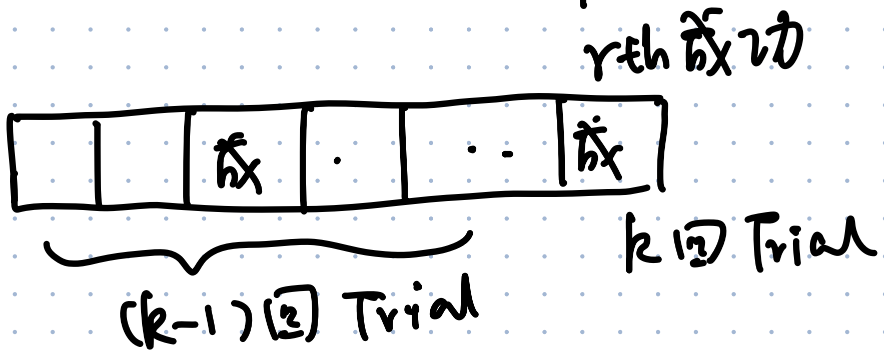
1回 $\boxed{X}$ Bernoulli Trial. $X = \begin{cases} 1 & (成) \quad p \\ 0 & (失) \quad q \end{cases}$

n回


$Y = k = \sum_{i=1}^{n} x_i$ = 成功回数 $\sim \binom{n}{k} p^k q^{n-k} \rightarrow Bin(n, p)$

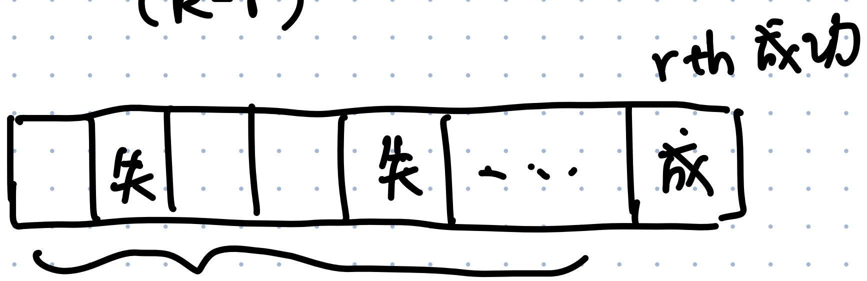
k回


$Z = k$ = Trial 総数 $\sim Geo(p)$

k回


$W = k$ = Trial 総数 $\sim \binom{k-1}{r-1} p^{r-1} q^{k-r} \cdot p \rightarrow NegBin(r, p)$

$\begin{cases} (r-1) \text{ 回 成功} \\ (k-1)-(r-1) \text{ 回失敗} \end{cases}$
$(k-r)$

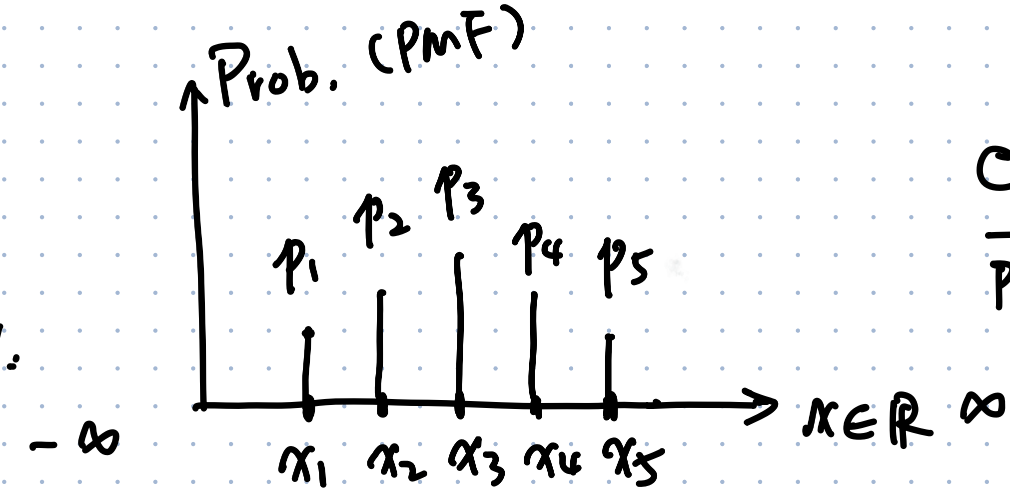


$V = k$ = 失敗数 $\sim \binom{k+r-1}{k} p^{r-1} q^k \cdot p. \rightarrow NegBin(r, p)$

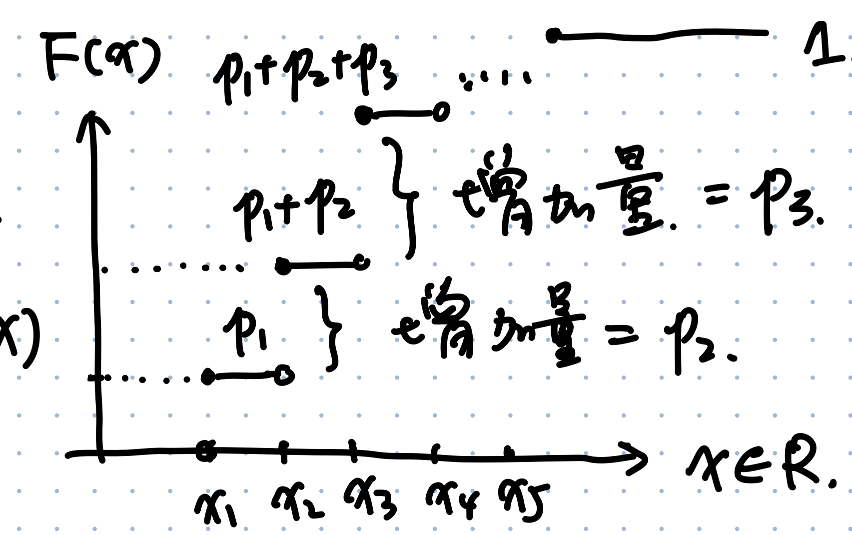
$\begin{cases} k \text{ 回 失敗} \\ (r-1) \text{ 成功} \end{cases}$
$(k+r-1)$回 Trial.

2.

Discrete RV.



CDF. $\xrightarrow[P(X\le x)]{}$



1

1.

0

F(x) = Jump する前の Prob.

{点}

↓

{线}

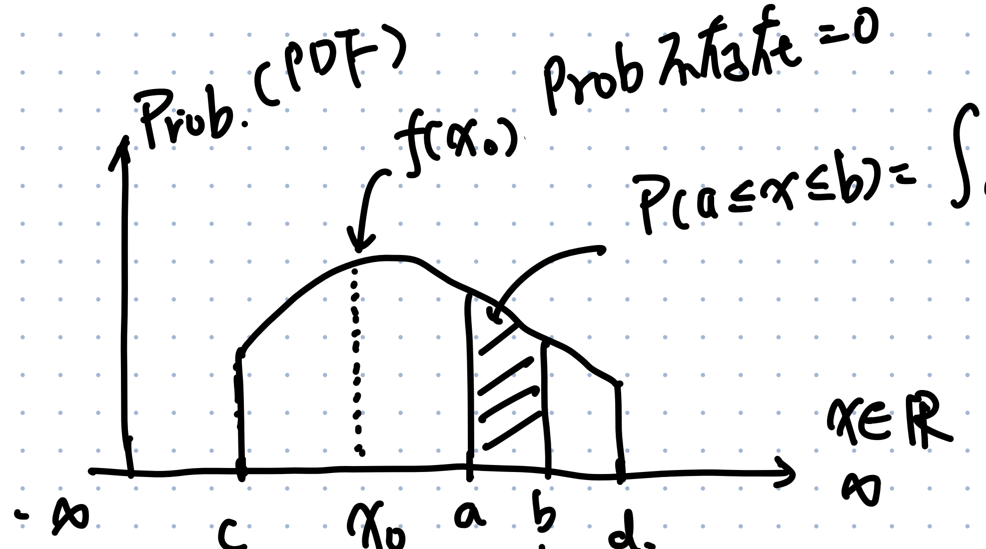
$$P(X=x_i)=\begin{cases} p_i & x_i\in\{x_1, x_2 \dots x_5\} \\ 0 & x_i\notin\{x_1 \dots x_5\}\end{cases} \quad \text{s.t. } \sum_{i=1}^{5} p_i = 1$$

Differential = Difference in x

dx

$$f(x)=\begin{cases} f(x) & x\in[c, d] \\ 0 & x\notin[c, d]\end{cases} \quad \text{s.t. } \int_c^d f(x)\,dx = \quad = 1.$$

Continuous RV.

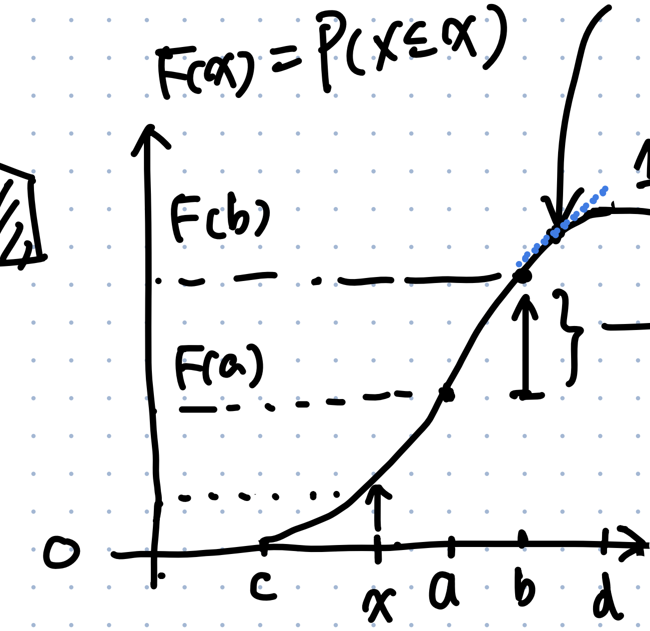


Prob 不存在 = 0

$$P(a\le x\le b) = \int_a^b f(x)\,dx = \frac{\quad}{1} = $$

面積 = 確率.

CDF →



$\frac{dF(x)}{dx} = f(x)$ = slope
speed
rate

増加量

$F(b)-F(a)$

$= \int_a^b f(x)\,dx$

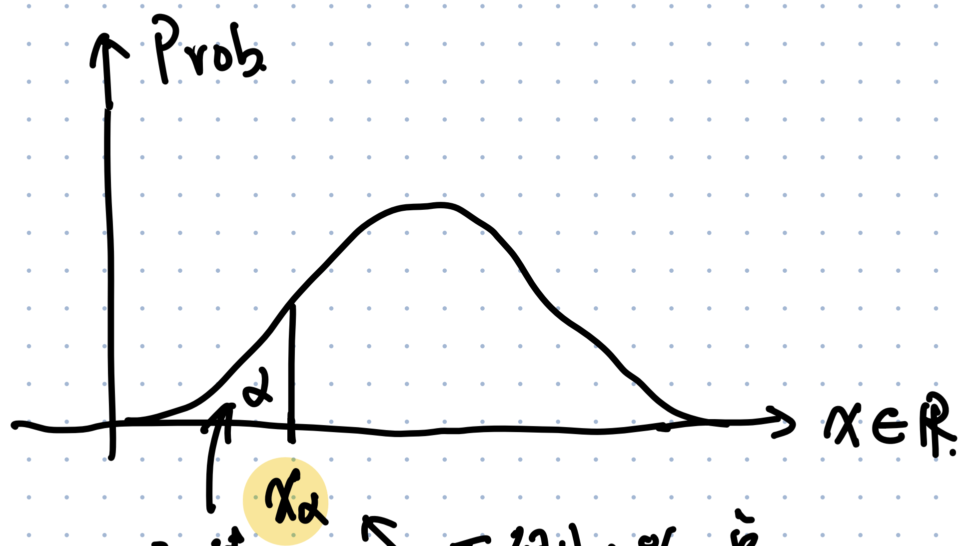
$$F(x) = P(X\le x)$$

$$= \int_{-\infty}^{x} f(t)\,dt = x \text{左側面積}$$

$$= \int_{-\infty}^{x} dF(t) = F(x) - \underbrace{F(-\infty)}_{=0} = \text{CDF 増加量}$$

Thm 2-5

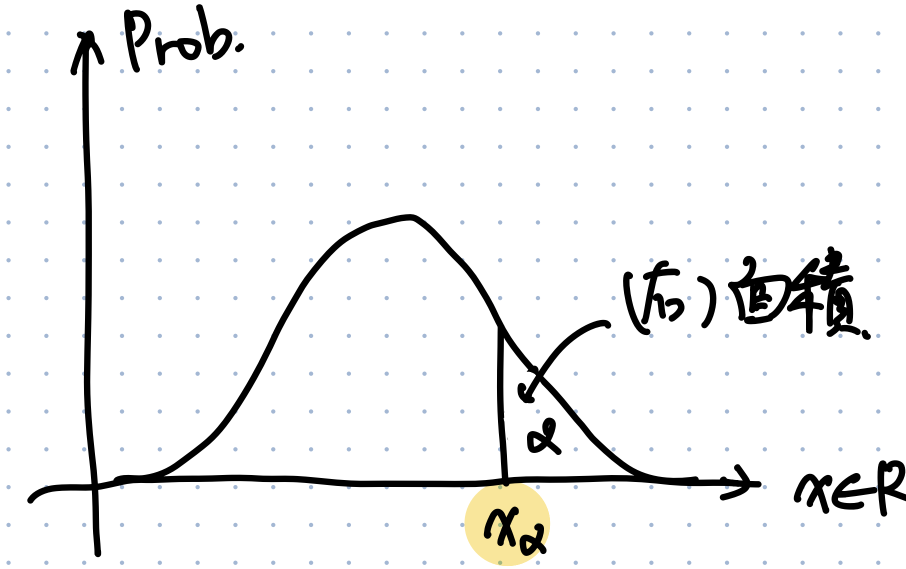
下側 α% 点



(左) 面積 $P(X \le x_\alpha) = \alpha$

下側 α% 点 (α 分位点)

上側 α% 点



(右) 面積 $P(X > x_\alpha) = \alpha$.

上側 α% 点

$$y = F(x)$$

$$\Updownarrow$$

$$F^{-1}(y) = x$$

Thm 2.8

$$\frac{d^k}{dt^k} M_X(t) = \frac{d^k}{dt^k} \int e^{tx} f_X(x)\, dx.$$

(連 & 微分可)

Leibnitz's Rule.

$$\frac{d^k}{dt^k} e^{tx} = x^k e^{tx}$$

$$= \int \frac{d^k}{dt^k}(e^{tx})\, f_X(x)\, dx$$

$$= \int x^k e^{tx} f_X(x)\, dx \overset{t=0}{=\!=} \int x^k f_X(x)\, dx = E[X^k]$$

$E[X^k]$ 求める方法:

① Def: $E[X^k] = \int x^k f(x)\, dx$

② MGF: $\frac{d^k}{dt^k} M_X(t) \Big|_{t=0}$

Def. 2.9. MGF of $N(\mu, \sigma^2)$

$$M_X(t) = E\{e^{tX}\} = \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} \exp\left\{tx - \left(\frac{x-\mu}{\sqrt{2}\,\sigma}\right)^2\right\} dx$$

Let $u = \frac{x-\mu}{\sqrt{2}\sigma}$ $\Rightarrow$ $x = \sqrt{2}u\sigma + \mu$.

$-\infty < x < \infty \Rightarrow u \in \left(\frac{-\infty-\mu}{\sqrt{2}\sigma}, \frac{+\infty-\mu}{\sqrt{2}\sigma}\right) = (-\infty, +\infty)$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{+\infty} \exp\{t(\underbrace{\sqrt{2}u\sigma+\mu}_{x}) - u^2\}\, \underbrace{d(\sqrt{2}u\sigma+\mu)}_{(\sqrt{2}\sigma\, du)}$$

uには関係ない

$$= \frac{\sqrt{2}\,\sigma}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{+\infty} \exp\{\sqrt{2}u\sigma t + \mu t - u^2\}\, du$$

$$= \frac{e^{\mu t}}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\{\underbrace{\sqrt{2}u\sigma t - u^2}_{-(u^2-\sqrt{2}u\sigma t+\frac{1}{2}\sigma^2t^2)+\frac{1}{2}\sigma^2t^2}\}\, du$$

$$= \frac{e^{\mu t}}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left\{-\left(u - \frac{\sqrt{2}}{2}\sigma t\right)^2 + \frac{1}{2}\sigma^2t^2\right\} du$$

uとは関係ない

$$= \frac{e^{\mu t + \frac{1}{2}\sigma^2t^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left\{-\left(u - \frac{\sqrt{2}}{2}\sigma t\right)^2\right\} du$$

Let $v = u - \frac{\sqrt{2}}{2}\sigma t \in (-\infty, +\infty)$

$u = v + \frac{\sqrt{2}}{2}\sigma t$, $du = dv$

$$= \frac{e^{\{\mu t + \frac{1}{2}\sigma^2t^2\}}}{\sqrt{\pi}} \underbrace{\int_{-\infty}^{\infty} \exp(-v^2)\, dv}_{= \sqrt{\pi}\ (*)}$$

$$= \exp\{\mu t + \tfrac{1}{2}\sigma^2 t^2\} = \exp\{\_\}$$

(*) Gaussian integral. (課題)

Thm. 2.11 (証.)

$M'(t) = \exp\{\_\} \cdot \{\mu + \sigma^2 t\}$

$M''(t) = \exp\{\_\}\{\mu + \sigma^2 t\}^2$

$\quad + \exp\{\_\}\{\sigma^2\}$

$E[X] = M'(0) = \mu$

$E[X^2] = M''(0) = \mu^2 + \sigma^2$

$Var[X] = E[X^2] - \{E[X]\}^2 = \sigma^2$

Thm 2.11

$$\left(\int_0^\infty e^{-\frac{z^2}{2}}\,dz\right)^2 = \int_0^\infty e^{-\frac{t^2}{2}}\,dt \cdot \int_0^\infty e^{-\frac{u^2}{2}}\,du$$

$u \perp t$ (独立)

Fubini Theorem

$$= \int_0^\infty \int_0^\infty e^{-\frac{t^2+u^2}{2}}\,dt\,du \quad \triangle$$

① → 2重積分

new

$$dd \to t = r\cos\theta$$
$$\to u = r\sin\theta$$

$$\Rightarrow t^2+u^2 = r^2\{(\cos\theta)^2+(\sin\theta)^2\} = r^2$$

→ ② t, u の座標を変更します

$$t>0,\ u>0 \Rightarrow \theta\in[0,\tfrac{\pi}{2}],\ r\in[0,+\infty)$$

$$dt\,du = J(r,\theta)\,dr\,d\theta = r\,dr\,d\theta$$

Jacobian

$$J(r,\theta) = \begin{vmatrix} \frac{\partial t}{\partial r} & \frac{\partial t}{\partial \theta} \\ \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \end{vmatrix} = \begin{array}{cc} A & B \\ \left|\begin{matrix} \frac{\partial r\cos\theta}{\partial r} & \frac{\partial r\cos\theta}{\partial\theta} \\ \frac{\partial r\sin\theta}{\partial r} & \frac{\partial r\sin\theta}{\partial\theta} \end{matrix}\right| \\ C & D \end{array} = AD - BC$$

$$= \cos\theta\cdot r\cdot\cos\theta + \sin\theta\cdot r\sin\theta$$

$$= r$$

$$\triangle = \int_0^{\frac{\pi}{2}} \underbrace{\int_0^\infty e^{-\frac{r^2}{2}}\cdot r\,dr}\,d\theta$$

$$= \frac{\pi}{2}\int_0^\infty r e^{-\frac{r^2}{2}}\,dr$$

$$= \frac{\pi}{2}\cdot\left[-e^{-\frac{r^2}{2}}\right]\Big|_0^\infty = \frac{\pi}{2}.$$

$$\Rightarrow \sqrt{\triangle} = \sqrt{\frac{\pi}{2}}$$

$$\int_a^b f(x)\,dx. \quad \text{Let } x = g(u).$$

$$u = g^{-1}(x) \in [g^{-1}(a), g^{-1}(b)]$$

(c, d.)

$$\int_a^b f(x)\,dx = \int_c^d f(g(u))\,d\,g(u)$$

$$= \int_c^d f(g(u))\,g'(u)\,du$$

new.

Integral Review.

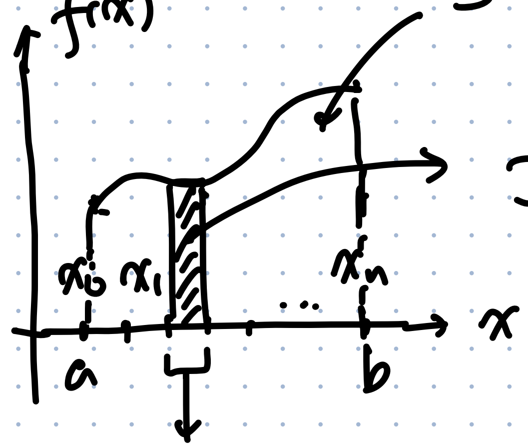
$\int_a^b f(x)\,dx = S$

(Riemann intergral)

$\lim_{\lambda\to 0}\left\{\sum_{i=1}^{n} f(x_i)\cdot\Delta x_i\right\}$

$\downarrow$

$\lambda = \max\{\Delta x_i\} \to 0$ ← Δx_i unequal.



$S = \int_a^b f(x)\,dx$.

$f(x_i)\cdot\Delta x_i = S_i$

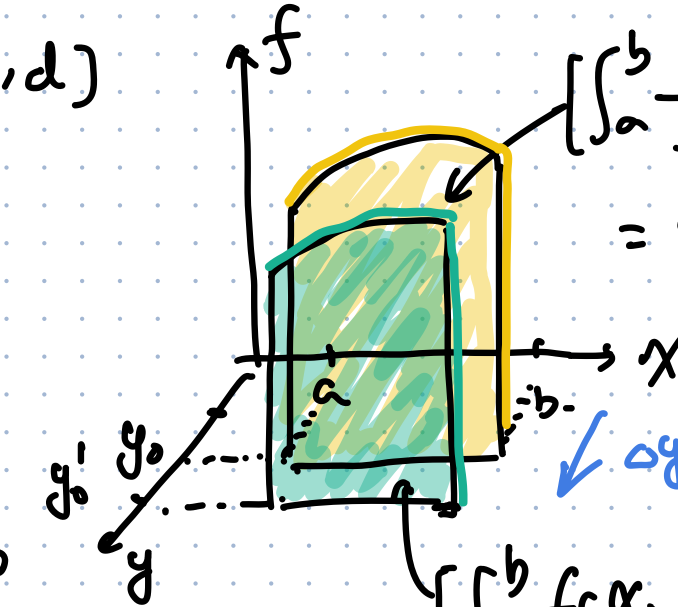
$\Delta i = \Delta x_i$

↑

长宽相等

Difference

① $\forall y_0 \in [c,d]$



$\left[\int_a^b f(x,y_0)\,dx\right]\cdot\Delta y_i = S_i\cdot\Delta y_i = V_i$ (体)

$\left[\int_a^b f(x,y_0')\,dx\right]\Delta y_i' = S_i'\cdot\Delta y_i' = V_i'$

$\max\{\Delta y_i\}\to 0$

$V = \lim_{\lambda_y\to 0}\sum_{i=1}^{n} S_i\,\Delta y_i = \int_c^d\left\{\int_a^b f(x,y)\,dx\right\}dy$.

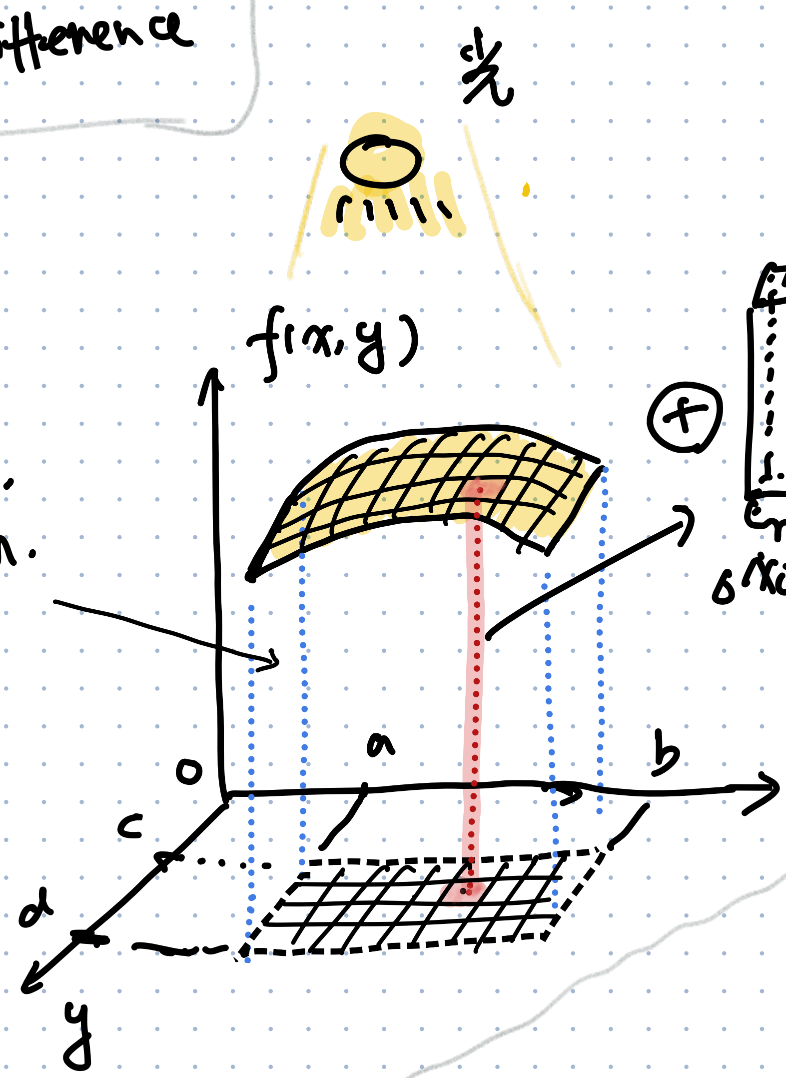
$\int_c^d \int_a^b f(x,y)\,dx\,dy = V$

$x\in[a,b]$

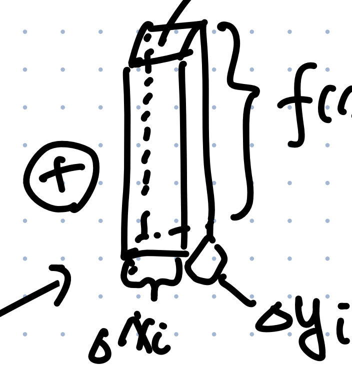
$y\in[c,d]$.

体積.

Volumn.



$\Delta x_i\cdot\Delta y_i = \Delta\sigma_i$

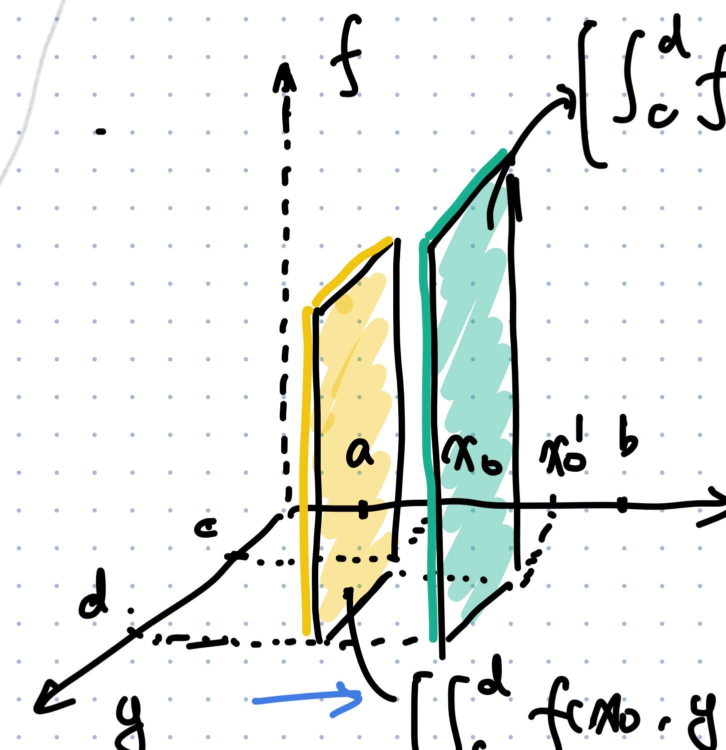


$= \lim_{\lambda\to 0}\sum_{i=1}^{n} f(x_i, y_i)\,\underbrace{\Delta x_i\Delta y_i}_{\Delta\sigma_i}$

$\downarrow$

$\max\{\boxed{\ }\}\to 0$

② $\forall x_0 \in [a,b]$



$\left[\int_c^d f(x_0', y)\,dy\right]\Delta x_i' = A_i'\cdot\Delta x_i' = \tilde{V}_i'$

(体)

$\left[\int_c^d f(x_0, y)\,dy\right]\Delta x_i = A_i\cdot\Delta x_i = \tilde{V}_i$

$\max\{\Delta x_i\}\to 0$

$V = \lim_{\lambda_x\to 0}\sum_{i=1}^{n} A_i\cdot\Delta x_i = \int_a^b\int_c^d f(x,y)\,dy\,dx$

Integral transformation.

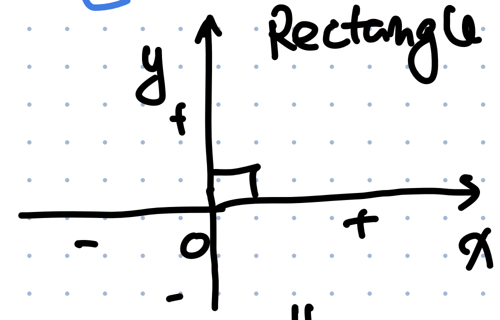
$dxdy = r\,dr\,d\theta$

$x>0,\ y>0$

$x = r\cos\theta = x(r,\theta) \quad r>0$

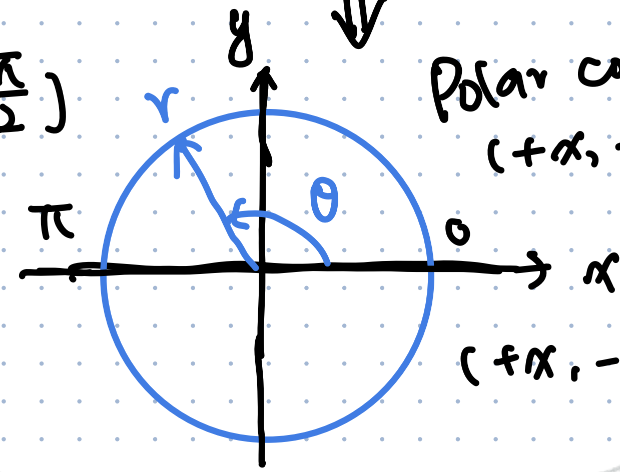
$y = r\sin\theta = y(r,\theta) \quad \theta \in [0, \frac{\pi}{2})$

Rectangle cordinates

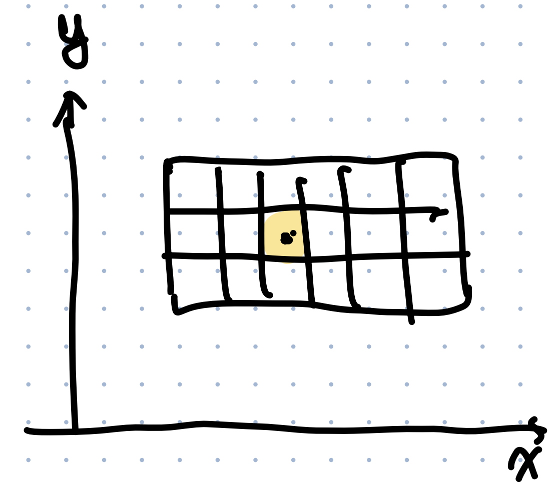


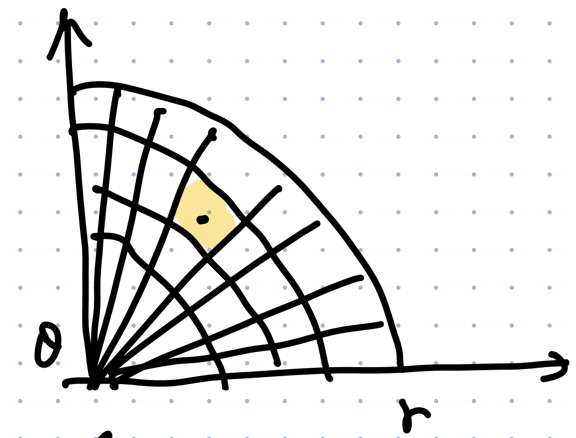
Polar coordinates

$(+x, +y)$



$(+x, -y)$





f f

高: $f(x,y)$

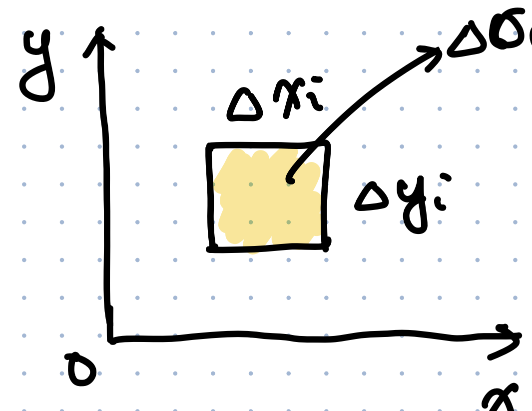
$\Downarrow$

$f(r\cos\theta, r\sin\theta) = g(r,\theta)$

底: $dxdy = \Delta x_i \cdot \Delta y_i = \Delta\sigma_i$

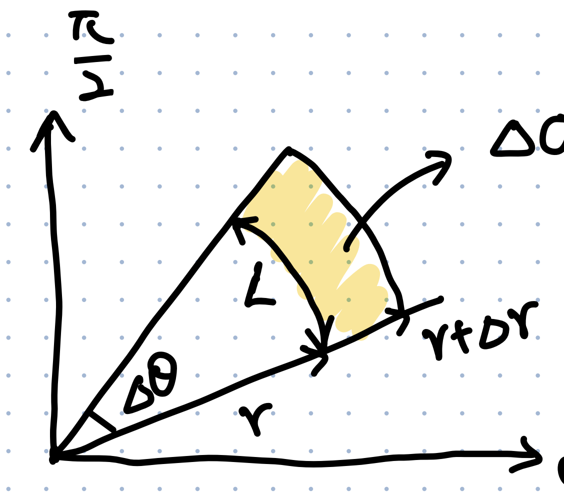
$\Downarrow$

Δa_i



$\Delta\sigma_i = dxdy$

$\longrightarrow$



$\Delta a_i = r \cdot d\theta \cdot dr$

$L = r \cdot \Delta\theta$

↳ 弧長. Distance between 2 points on a curve.

arclength :

積分 $= V = \lim_{\lambda \to 0} \sum_i S_i \cdot H_i$

↑底 ↑高

$\frac{dxdy}{drd\theta}$ $\frac{\cancel{\Delta\sigma_i}}{\cancel{\Delta a_i}} = \text{rate} = r = A.$

↑

変化比率.

$\iint_{\square} f(x,y)\,dxdy$

$= \iint_{\triangledown} g(r,\theta)\,A \cdot dr \cdot d\theta$

$J(r,\theta)$

↳ Jacobian $= \det\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$

Determinant of Jacobian Matrix.

(後.代)

① 1重: $x = \tilde{x}(u)$ $\quad dx = \frac{d\tilde{x}}{du}du$

② 多重: $x = \tilde{x}(u,v,w\cdots)$ $\quad dxdydz\cdots$

$y = \tilde{y}(u,v,w\cdots)$ $\quad = A\,du\,dv\,dw$

$z = \tilde{z}(u,v,w\cdots)$ $\quad \downarrow$

$\vdots$ $\quad J(u,v,w\cdots)$

Normal $\approx$ Binomial

Thm 2.14

$X \sim N(\mu, \sigma^2)$: $M_X(t) = \exp\{\mu t + \frac{\sigma^2 t^2}{2}\}$

$Y \sim Bin(n, p)$: $M_Y(t) = \{(1-p) + pe^t\}^n$.

$\mu = np$

$\sigma = n(p)(1-p)$

$$1-p+pe^t = 1-p+p(1+t+\frac{t^2}{2!}+\cancel{\frac{t^3}{3!}}+O(t^3))$$

$$= 1+pt+\frac{p}{2}t^2+O(t^3)$$

$\exists M>0.$

Big O. : $|(1-p+pe^t)| \le M|t^3|$.

as. $t \to 0$.

$\downarrow$

Boundary

(境界)

$$(1+pt+\frac{pt^2}{2}+O(t^3))^n = \exp\{n\log(1+pt+\frac{pt^2}{2}+O(t^3))\}$$

$$= \exp\{n\{pt+\frac{pt^2}{2}-\frac{p^2t^2}{2}+O(t^3)\}\}.$$

$$= \exp\{\overset{\mu}{np}t+\frac{\overset{\sigma^2}{np(1-p)}t^2}{2}+O(t^3 n)\}$$

$\downarrow$ for $t = \frac{s}{\sqrt{n}} \to 0$

0

(詳.略?)

$n \to \infty$

$\longrightarrow M_X(t)$

$\log(1+y) = y - \frac{y^2}{2} + \cdots\cdots$

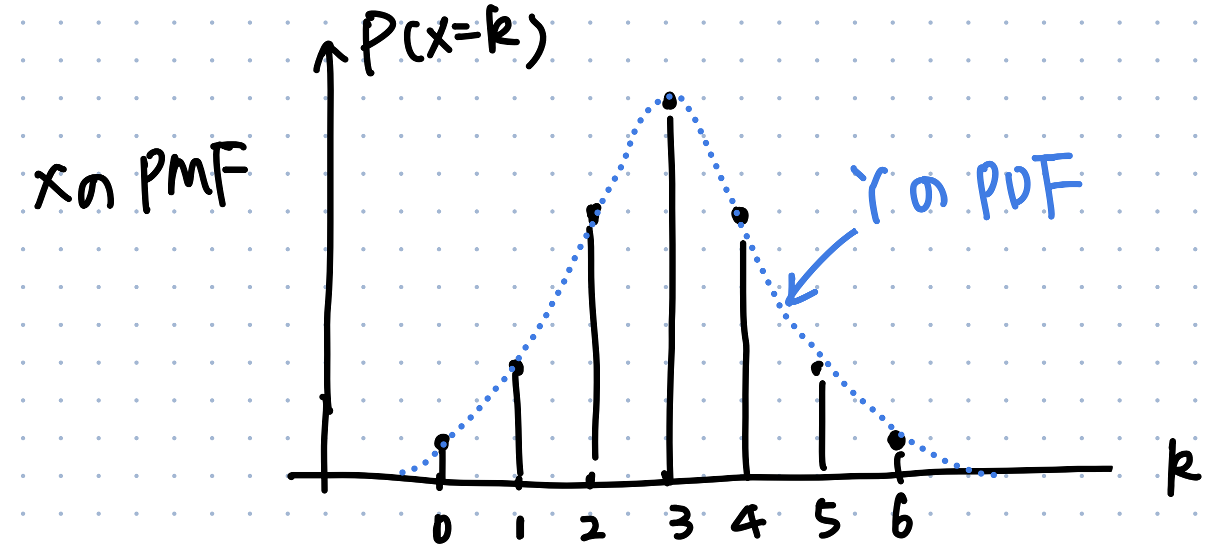
$$\log(1+\underbrace{pt+\frac{pt^2}{2}+O(t^3)}_{y})$$

$$= pt+\frac{pt^2}{2}+O(t^3) - \frac{(pt+\frac{pt^2}{2}+O(t^3))^2}{2}.$$

$$= pt+\frac{pt^2}{2}-\frac{p^2t^2}{2}+O(t^3)$$

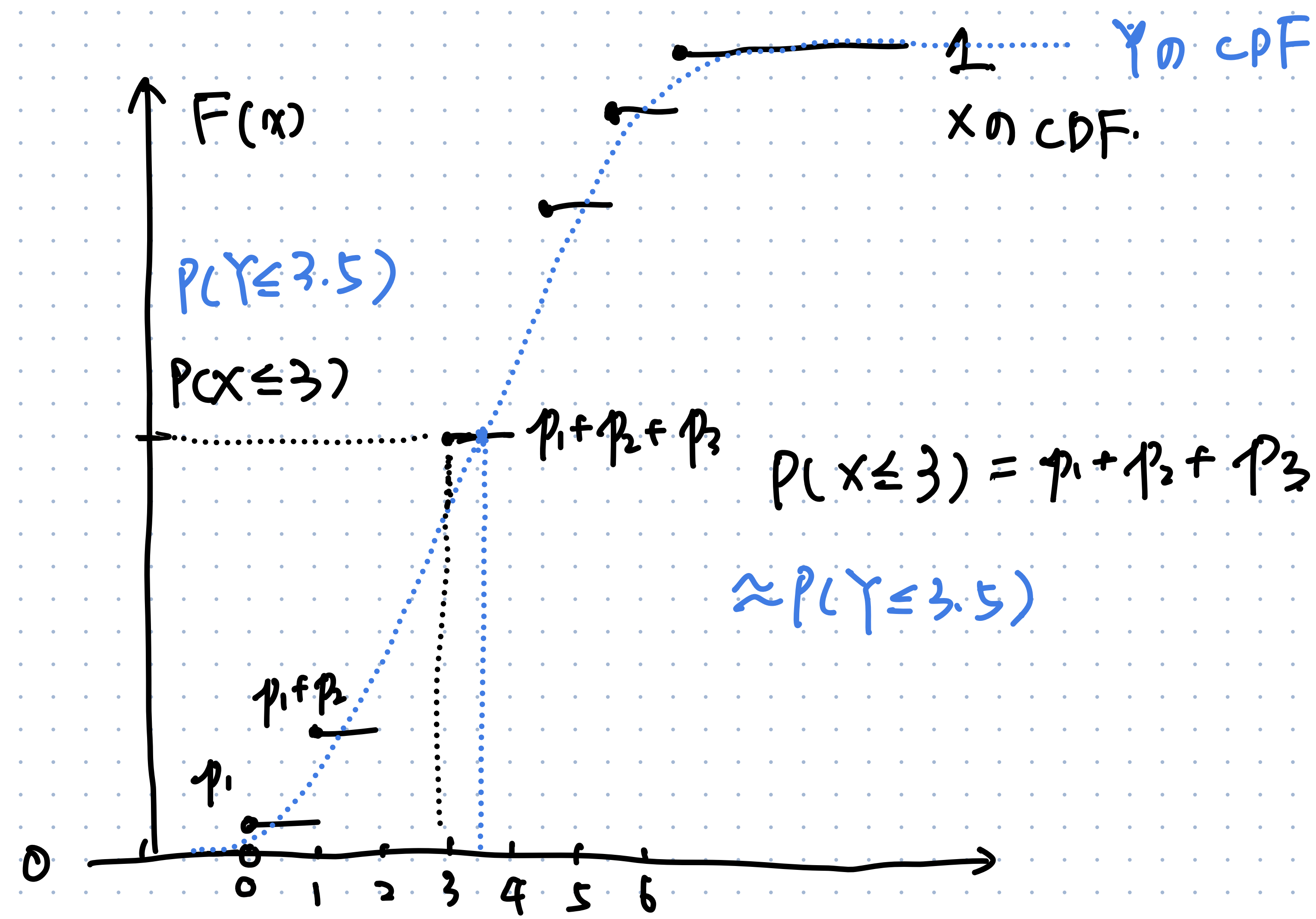
Continuity Correction

連続性補正



$X \sim Bin(n, p)$

$Y \sim N(\mu = np,\ \sigma^2 = np(1-p))$



$P(X \leq 3) = p_1 + p_2 + p_3$

$\approx P(Y \leq 3.5)$

Def. 2.15.

$$M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx$$

$$= \int_a^b e^{tx} \frac{1}{b-a}\,dx$$

$$= \frac{e^{tx}}{(b-a)t}\Big|_a^b \quad \circledast$$

$t \neq 0$. $\quad M_X(t) = \dfrac{e^{tb} - e^{ta}}{(b-a)t}$

$t = 0$: $\quad E[e^{0X}] = E[1] = 1$

$$M_X(t) = \begin{cases} 1 & t = 0 \\ \dfrac{e^{tb} - e^{ta}}{(b-a)t} & t \neq 0 \end{cases}$$

9

$$M'(t) = \frac{(be^{tb} - ae^{ta})\cdot(b-a)t - (e^{tb} - e^{ta})(b-a)}{(b-a)^2 t^2}$$

$$m'(0) \overset{t=0}{=\!=} \frac{0}{0} \quad ??$$

① L'Hopital's Rule. $\rightarrow \left(\frac{0}{0}.\ \frac{+\infty}{+\infty}.\ \frac{-\infty}{-\infty} \right)$

$$M'(t) = \lim_{t\to 0} \frac{f(x)}{g(x)} = \lim_{t\to 0} \frac{f'(x)}{g'(x)}$$

$$= \lim_{t\to 0} \frac{(be^{tb} - ae^{ta})t - (e^{tb} - e^{ta})}{(b-a)t^2}$$

L'Hopital.
$$= \lim_{t\to 0} \frac{(b^2e^{tb} - a^2e^{ta})\cdot t + \cancel{(be^{tb} - ae^{ta})} - \cancel{(be^{tb} - ae^{ta})}}{2t(b-a)} \qquad \frac{0}{0}$$

L'Hopital
$$= \lim_{t\to 0} \frac{(b^3e^{tb} - a^3e^{ta})\cdot t + (b^2e^{tb} - a^2e^{ta})}{2(b-a)}$$

$$= \frac{b^2 - a^2}{2(b-a)} = \frac{b+a}{2}$$

Ex 2.18.

$$P(X>s+t \mid X>s) = \frac{P(X>s+t,\ X>s)}{P(X>s)} = \frac{P(X>s+t)}{P(X>s)} = \frac{1-F(s+t)}{1-F(s)} \quad \triangle$$

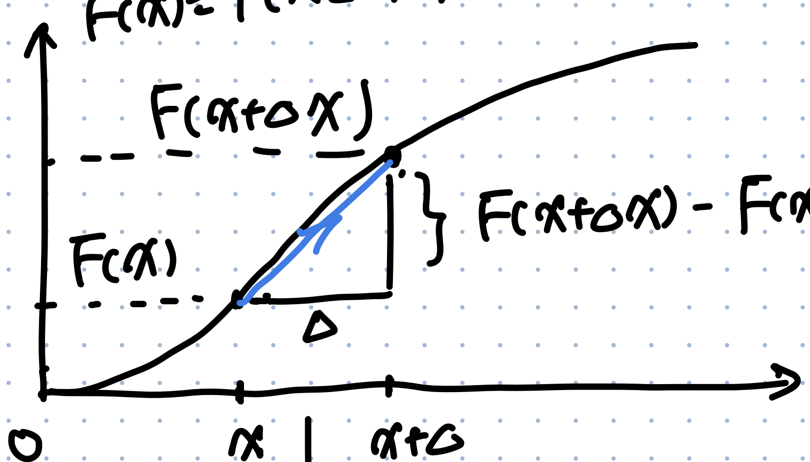
$$\triangle = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = 1-F(t) = P(X>t)$$

$\forall\ x \geq 0.\quad f(x) = \lambda e^{-\lambda x}$

$$F(x) = P(X \leq x) = \int_0^x \lambda e^{-\lambda t}\,dt = -e^{-\lambda t}\Big|_0^x = -e^{-\lambda x}+1 \Rightarrow 1-F(x) = e^{-\lambda x}.$$

Thm 2.19

$F(x) = P(X \leq x)$



$\} \ F(x+\Delta x) - F(x) = \Delta F(x)$

$\Delta \to 0$

$$\underbrace{\lim_{\Delta \to 0} \frac{\Delta F(x)}{\Delta}}_{\text{x点の Death Speed./Slope. Rate}} \cdot \frac{1}{1-F(x)} = \frac{\text{死亡の Speed}}{P(X>x)} \to \text{Hazard.}$$

$\Delta \neq 0.$

if $\Delta = 0.\ P(X=x)=0$

x点仍生存のProb.

$S(t)$

望むこと

x点仍生存する条件の下で x の瞬間死亡の可能性.

望んでないこと.

$$\int_0^x \overbrace{\frac{f(t)}{1-F(t)}}^{h(t)}\,dt = -\log(1-F(t))\Big|_0^x$$

$$\Rightarrow \int_0^x h(t)\,dt = -\log(1-F(x))$$

$$-\exp \int_0^x h(t)\,dt = 1-F(x) \Rightarrow F(x) = \cdots$$

$$f(x) = h(t)[1-F(x)] = h(t)\{\ .\ \_\ .\ \}$$

$f(x)$ = 死亡の確率

$h(x)$ = 死亡の条件付き確率

Thm. 2.21

$$\Gamma(\alpha+1) = \int_0^\infty t^\alpha e^{-t} dt.$$

$$t = \lambda \cdot u \Rightarrow e^{-t} = e^{-(\lambda u)} \qquad u \in [0, \infty)$$

$$dt = \lambda du$$

$$t^\alpha = \lambda^\alpha \cdot u^\alpha.$$

$$\Gamma(\alpha+1) = \int_0^\infty \lambda^\alpha u^\alpha \cdot e^{-\lambda u} \cdot \lambda du$$

$$= \lambda^{\alpha+1} \int_0^\infty u^\alpha e^{-\lambda u} du$$

$$= \lambda^{\alpha+1} \int_0^\infty u^\alpha \overbrace{d e^{-\lambda u} (-\lambda)^{-1}}^{e^{-\lambda u} du} \rightarrow \int u dv = uv - \int v du$$

$$= -\lambda^\alpha \left[\overset{0}{u^\alpha e^{-\lambda u} \Big|_0^\infty} - \int_0^\infty e^{-\lambda u} du^\alpha. \right]$$

$$= \lambda^\alpha \int_0^\infty e^{-\lambda u} \alpha \cdot u^{\alpha-1} du$$

$$= \alpha \int_0^\infty e^{-\lambda u} (\lambda u)^{\alpha-1} d\lambda u \qquad \lambda u = t.$$

$$= \alpha \int_0^\infty e^{-t} t^{\alpha-1} dt = \alpha \Gamma(\alpha)$$

$$\Gamma(n) = (n-1) \cdot \Gamma(n-1)$$

$$= (n-1)(n-2) \cdot \Gamma(n-2)$$

$$= \cdots = (n-1)!$$

$$\Gamma(1) = \int_0^\infty t^1 e^{-t} dt$$

$$= \int_0^\infty t d e^{-t}$$

$$= t e^{-t} \Big|_0^\infty - \int_0^\infty e^{-t} dt.$$

$$= -\left[-e^{-t} \Big|_0^\infty \right] = 1$$

$\Gamma(\frac{1}{2}) = \sqrt{\pi}$. (課題)

11

Def. 2.22. MGF of Gamma Distribution.

12

$$M_X(t) = E[e^{tX}] = \int_0^\infty e^{tx} \cdot \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}} \, dx$$

$$t - \frac{1}{\beta} = -\frac{1}{u} \Rightarrow \frac{1}{u} = \frac{t\beta - 1}{\beta} \Rightarrow u = \frac{\beta}{t\beta-1}$$

$$= \int_0^\infty e^{x(t-\frac{1}{\beta})} x^{\alpha-1} \frac{1}{\Gamma(\alpha)(t-\frac{1}{\beta})^\alpha} \cdot \frac{(t-\frac{1}{\beta})^\alpha}{\beta^\alpha} \, dx$$

$$\frac{1}{\beta} = t - \frac{1}{u}$$

$$= \int_0^\infty e^{x(-\frac{1}{u})} x^{\alpha-1} \frac{1}{\Gamma(\alpha)(-\frac{1}{u})^\alpha} \cdot (-\frac{1}{u})^\alpha \cdot (t-\frac{1}{u})^\alpha \, dx$$

$$(\frac{1}{\beta})^\alpha = (t - \frac{1}{u})^\alpha$$

$$= \int_0^\infty e^{-\frac{x}{u}} \cdot x^{\alpha-1} \frac{1}{\Gamma(\alpha)} \cdot \frac{1}{u^\alpha} \cdot u^\alpha \cdot (t-\frac{1}{u})^\alpha \, dx$$

$$= u^\alpha \cdot (t-\frac{1}{u})^\alpha \underbrace{\int_0^\infty e^{-\frac{x}{u}} x^{\alpha-1} \frac{1}{\Gamma(\alpha)u^\alpha} \, dx}_{\int_0^\infty \text{Gamma}(\alpha, u) \, dx = 1}$$

$$= \frac{\beta^\alpha}{(t\beta-1)^\alpha} \cdot \frac{1}{\beta^\alpha}$$

$$= (1-t\beta)^{-\alpha}$$

$$E[X] = M'(t) = -\alpha(1-t\beta)^{-\alpha-1}(-\beta) \xrightarrow{t=0} M'(0) = (-\alpha)(-\beta) = \alpha\beta$$

$$E[X^2] = M''(0) = \cdots$$

Thm. 2.23.

$X \sim \text{Gamma}(\alpha, \beta)$

$$P(X \le x) = \int_0^x \frac{1}{\Gamma(\alpha)\beta^\alpha} t^{\alpha-1} e^{-\frac{t}{\beta}}\, dt$$

$\alpha \in \mathbb{Z}$

$$= \frac{1}{(\alpha-1)!\,\beta^\alpha} \int_0^x t^{\alpha-1} \underbrace{e^{-\frac{t}{\beta}}\, dt}$$

$$= \frac{1}{(\alpha-1)!\,\beta^\alpha} \int_0^x t^{\alpha-1}\, d e^{-\frac{t}{\beta}} (-\beta) \qquad \left(\int_0^x (-\beta) e^{-\frac{x}{\beta}}\, dt^{\alpha-1} \right)$$

$$= \frac{1}{(\alpha-1)!\,\beta^\alpha} \left\{ t^{\alpha-1}(-\beta) e^{-\frac{t}{\beta}} \Big|_0^x - \int_0^x (-\beta) e^{-\frac{t}{\beta}} (\alpha-1) t^{\alpha-2}\, dt \right\}$$

$$= \frac{1}{(\alpha-1)!\,\beta^\alpha} \left\{ x^{\alpha-1} (-\beta) e^{-\frac{x}{\beta}} + \beta(\alpha-1) \int_0^x e^{-\frac{t}{\beta}} t^{\alpha-2}\, dt \right\}$$

$$= \underbrace{\frac{-1}{(\alpha-1)!\,\beta^{\alpha-1}} x^{\alpha-1} e^{-\frac{x}{\beta}}}_{*} + \boxed{\frac{1}{(\alpha-2)!\,\beta^{\alpha-1}} \int_0^x e^{-\frac{t}{\beta}} t^{\alpha-2}\, dt} \quad \triangle$$

$Y \sim \text{Poisson}(\lambda = \frac{x}{\beta})$

$$P(Y=k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

$$P(Y=\alpha-1) = \frac{e^{-\lambda}\lambda^{\alpha-1}}{(\alpha-1)!}$$

$$= \frac{e^{-\frac{x}{\beta}} \left(\frac{x}{\beta}\right)^{\alpha-1}}{(\alpha-1)!}$$

$$= \frac{e^{-\frac{x}{\beta}} x^{\alpha-1}}{(\alpha-1)!\,\beta^{\alpha-1}}$$

$$= -*$$

$P(X \le x)$

$$\triangle = \boxed{} - P(Y=\alpha-1)$$

$$\vdots \quad \int_0^x \text{Gamma}(\alpha-1, \beta)\, dt - P(Y=\alpha-1)$$

$$\vdots$$

$$= \int_0^x \text{Gamma}(1, \beta)\, dt \leftarrow 1 - P(Y=0)$$

$$- \{ P(Y=\alpha-1) + P(Y=\alpha-2) + \cdots P(Y=1) \}$$

Ref. 2.27

$$M_X(t) = E[e^{tX}] = \int_0^1 e^{tx} \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1}\, dx$$

$$= \frac{1}{B(\alpha, \beta)} \int_0^1 e^{tx} x^{\alpha-1}(1-x)^{\beta-1}\, dx$$

$$\frac{d^n M_X(t)}{dt^n} = \frac{1}{B(\alpha, \beta)} \int_0^1 \frac{d^n}{dt^n} \cdot e^{tx} x^{\alpha-1}(1-x)^{\beta-1}\, dx$$

$$= \frac{1}{B(\alpha, \beta)} \cdot \int_0^1 x^n e^{tx} \cdot x^{\alpha-1}(1-x)^{\beta-1}\, dx$$

$$\overset{t=0}{=} \frac{1}{B(\alpha, \beta)} \int_0^1 x^n \cdot x^{\alpha-1}(1-x)^{\beta-1}\, dx$$

$$= \frac{1}{B(\alpha, \beta)} \int_0^1 \underbrace{x^{n+\alpha-1}(1-x)^{\beta-1}\, dx}$$

$$= \frac{B(n+\alpha, \beta)}{B(\alpha, \beta)}$$

2変量確率変数 (X, Y)

Prob. mass function:

① 同時PMF $\rightarrow P(X=x_i, Y=y_j)$

② 周辺PMF.

$\rightarrow \sum_{y \in \mathcal{Y}} P(X=x_i, Y=y) = P(X=x_i)$

$\sum_{x \in \mathcal{X}} P(X=x, Y=y_j) = P(Y=y_j)$ (R.V.) X

③ 条件付きPMF.

$P(X=x_i \mid Y=y_j)$

$= \dfrac{P(X=x_i, Y=y_j)}{P(Y=y_j)} = \dfrac{\text{同時 PMF}}{\text{周辺 PMF}}$

$\sum_{x_i \in \mathcal{X}} P(X=x_i \mid Y=y_j) = 1$

$= \dfrac{P(x_1 \mid y_j)}{P(y_j)} + \dfrac{P(x_2 \mid y_j)}{P(y_j)} + \cdots \dfrac{P(x_i \mid y_j)}{P(y_j)}$

$= \dfrac{P(y_j)}{P(y_j)} = 1$

Y ③ $P(X=x_i \mid Y=y_j)$ $X=x_1$の周辺PMF. ②

	y_1	y_2	···	y_j	···	Marginal total.
(実現値) x_1	$P(x_1, y_1)$	$P(x_1, y_2)$				$P(x_1, y_1) + P(x_1, y_2) + \cdots\cdots P(x_1, y_j) + \cdots = P(X=x_1)$
x_2	$P(x_2, y_1)$	$P(x_2, y_2)$				
⋮ x_i				① $P(x_i, y_j)$		$P(X=x_i)$
⋮						
Marginal total. ②	$P(y_1, x_1) + P(y_1, x_2) + \cdots + P(y_1, x_i) + \cdots = P(Y=y_1)$			$P(Y=y_j)$		

$P(Y=y_j \mid X=x_i)$

$Y=y_1$ の周辺PMF

②

$E\{\underbrace{E[\underbrace{g(X,Y)|X}]}\}$ $X \in \mathcal{X} = \{x_1, x_2, \dots\}$

$Y \in \mathcal{Y} = \{y_1, y_2, \dots\}$

固定した条件. (条件付き)

変数 → Yの期待値を求める

$\boxed{E[g(X,Y)|X]} = \sum_{y \in \mathcal{Y}} g(x,y) \cdot \dfrac{P(X=x, Y=y)}{P(X=x)}$

$g(X=x, Y=y_2) \to g(X)$

$= \underline{g(x\ y_1)\dfrac{P(x,y_1)}{P(x)}} + \boxed{g(x,y_2)\dfrac{P(x,y_2)}{P(x)}} + \cdots + \underline{g(x,y_j)\cdot\dfrac{P(x,y_j)}{P(x)}} + \cdots$

$E\{\boxed{}\} = E\left[g(x\ y_1)\cdot\dfrac{P(x\ y_1)}{P(x)}\right] + E\left[g(x\ y_2)\dfrac{P(x\ y_2)}{P(x)}\right] + \cdots + \cdots$

変数　$Y=y_1$:固定した実現値

① ② ⓘ

$= \underline{\sum_{x=0}^{\infty} g(x\ y_1)\cdot\dfrac{P(x\ y_1)}{P(x)}}\cdot P(x) + \sum_{x=0}^{\infty}\underline{g(x\ y_2)\dfrac{P(x\ y_2)}{P(x)}}\cdot P(x) + \cdots + \sum_{x=0}^{\infty}$

① $g(x_1\ y_1)\cdot\dfrac{P(x_1\ y_1)}{P(x_1)}\cdot P(x_1) + g(x_2\ y_1)\cdot\dfrac{P(x_2\ y_1)}{P(x_2)}\cdot P(x_2) + \cdots$

$+$

② $g(x_1\ y_2)\cdot\dfrac{P(x_1\ y_2)}{P(x_1)} P(x_1) + g(x_2\ y_2)\cdot\dfrac{P(x_2\ y_2)}{P(x_2)}\cdot P(x_2) + \cdots$

$+$

$\vdots$

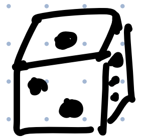
$+$

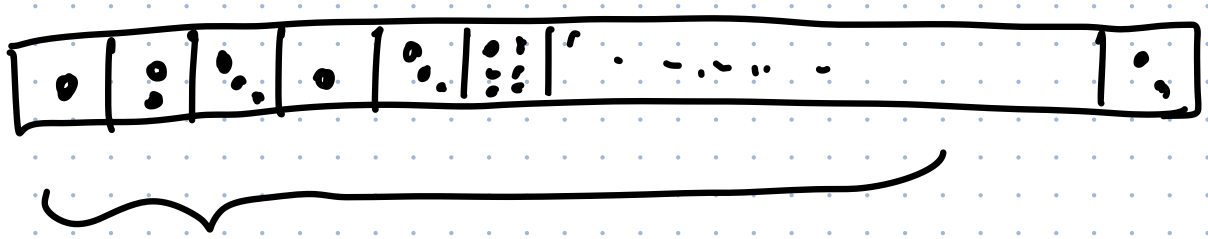
ⓘ $g(x_1, y_j)\cdot\dfrac{P(x_1, y_j)}{P(x_1)} P(x_1) + g(x_2, y_j)\cdot\dfrac{P(x_2, y_j)}{P(x_2)}\cdot P(x_2) + \cdots$

(略記)

$= \sum_{x=0}^{\infty}\sum_{y=0}^{\infty} g(x,y)\cdot P(x,y) = E[g(X,Y)]$

Multinomial Distribution 多項分布





n回投げる

$X_1 = x_1$ 回
$X_2 = x_2$ 回
$X_3 = x_3$ 回
$\vdots$
$X_6 = x_6$ 回

$$\left.\begin{array}{l} X_1 = x_1 \\ X_2 = x_2 \\ X_3 = x_3 \\ \vdots \\ X_6 = x_6 \end{array}\right\} x_1 + x_2 + \cdots + x_6 = n \text{ 回}$$

X_1: ⚀が出る回数 ⇒ #⚀　　$P($⚀出$) = p_1$

X_2: #⚁　　$P($⚁出$) = p_2$

X_3: #⚂　　$\vdots$

$\vdots$

X_6: #⚅　　$P($⚅$) = p_6$

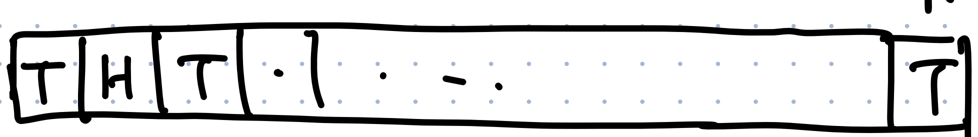
$p_1 + p_2 + \cdots + p_6 = 1$

$$P(X_1 = x_1, X_2 = x_2, \ldots, X_6 = x_6) = \binom{n}{x_1}\binom{n-x_1}{x_2}\binom{n-x_1-x_2}{x_3}\cdots \; p_1^{x_1} p_2^{x_2} \cdots p_6^{x_6}$$

$$= \frac{n!}{x_1!\, x_2! \cdots x_6!} p_1^{x_1} \cdots p_6^{x_6} \quad \text{(多項分布)}$$

Ⓣ
Ⓗ

n回



X_1: Tが出る回数 ⇒ #T　　$P(T) = p_1 \Rightarrow P(H) = 1 - p_1$
$= p_2$

X_2: #H $= n - X_1$

$$P(X_1 = x_1, X_2 = n - x_1) = \binom{n}{x_1} \cdot p_1^{x_1} (1-p_1)^{n-x_1}$$

(二項分布)

4. (離散)

①

Suppose: $X \sim$ Discrect Uniform Dist. $\begin{cases} P(X=x_1)=\frac{1}{2} \\ P(X=x_2)=\frac{1}{2} \end{cases}$

$Y \sim$ Discrect Uniform Dist. $\begin{cases} P(Y=y_1)=\frac{1}{2} \\ P(Y=y_2)=\frac{1}{2} \end{cases}$

$X \perp\!\!\!\perp Y$

(X,Y): $f_{X,Y}(x,y) = P(X=x, Y=y) = \frac{1}{4}$

$P(x_1,y_1) = P(x_2,y_1) = P(x_1,y_1) = P(x_1,y_2) = \frac{1}{4}$

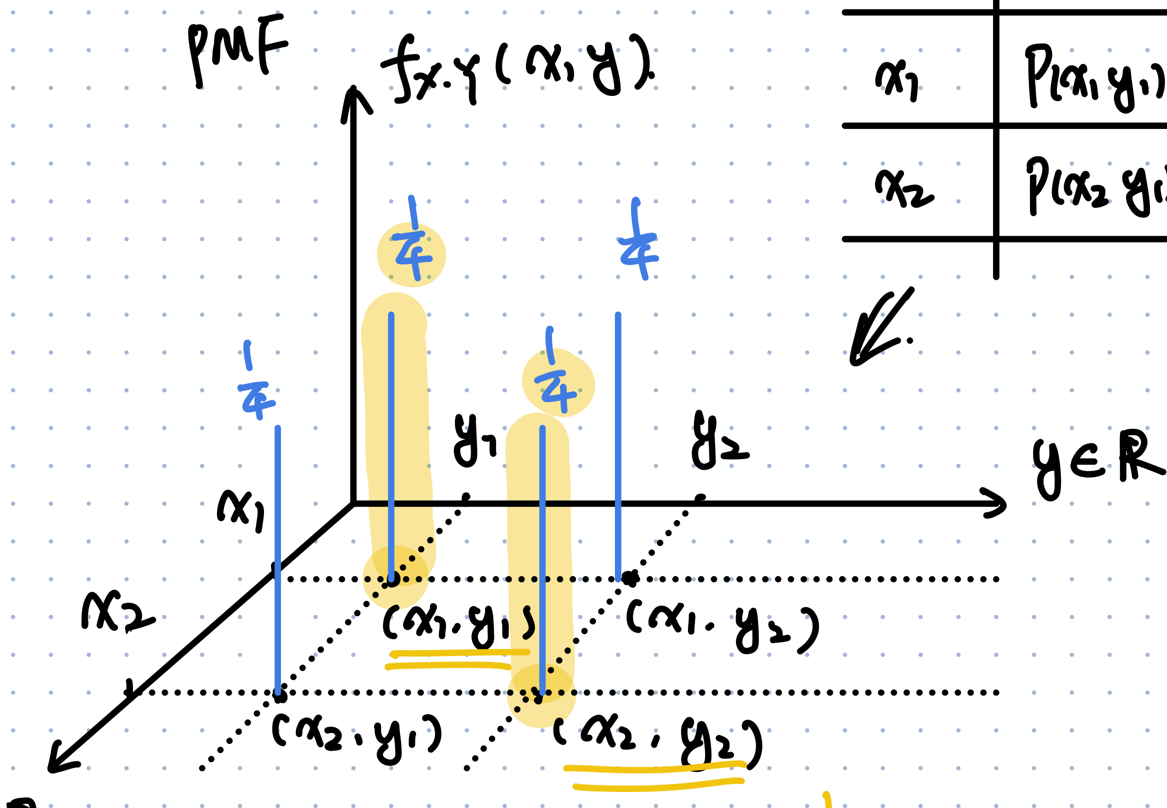
X: $x \in \mathcal{X} = \{x_1, x_2\}$

Y: $y \in \mathcal{Y} = \{y_1, y_2\}$.

分割表.

	y_1	y_2
x_1	$P(x_1,y_1)$	$P(x_1,y_2)$
x_2	$P(x_2,y_1)$	$P(x_2,y_2)$

PMF



$C = \{(x_1,y_1), (x_2,y_2)\}$

$\xrightarrow{\text{連續}}$

$X \sim$ Cont. Uniform $[x_1, x_2]$. $f_X(x) = \frac{1}{x_2 - x_1}$

$Y \sim$ cont. uniform $[y_1, y_2]$ $f_Y(y) = \frac{1}{y_2 - y_1}$

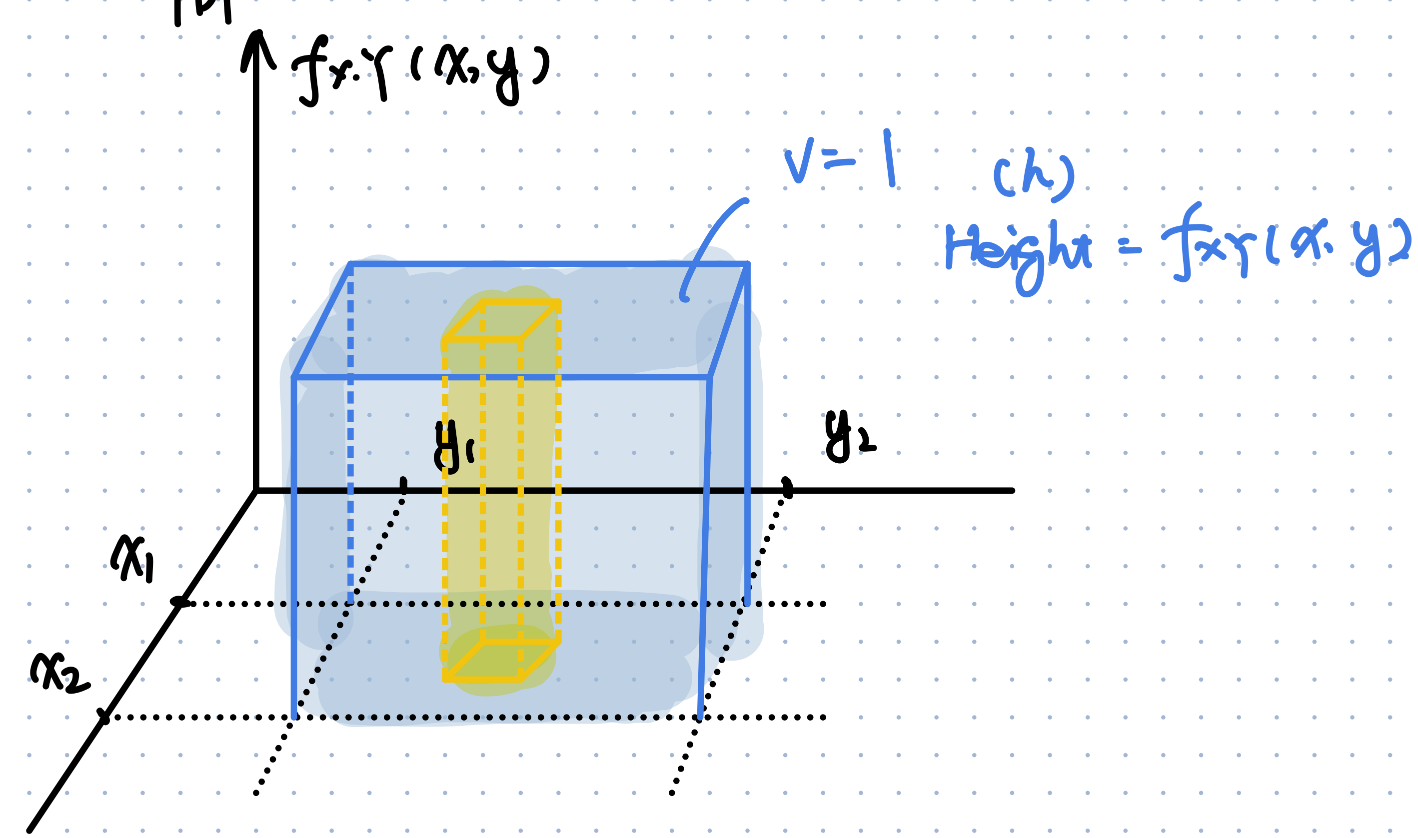
$X \perp\!\!\!\perp Y$.

(X,Y): $f_{X,Y}(x,y) = \frac{1}{(x_2 - x_1)} \cdot \frac{1}{y_2 - y_1}$

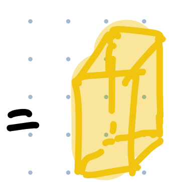
X: $x \in \mathcal{X} = [x_1, x_2]$

Y: $y \in \mathcal{Y} = [y_1, y_2]$

PDF

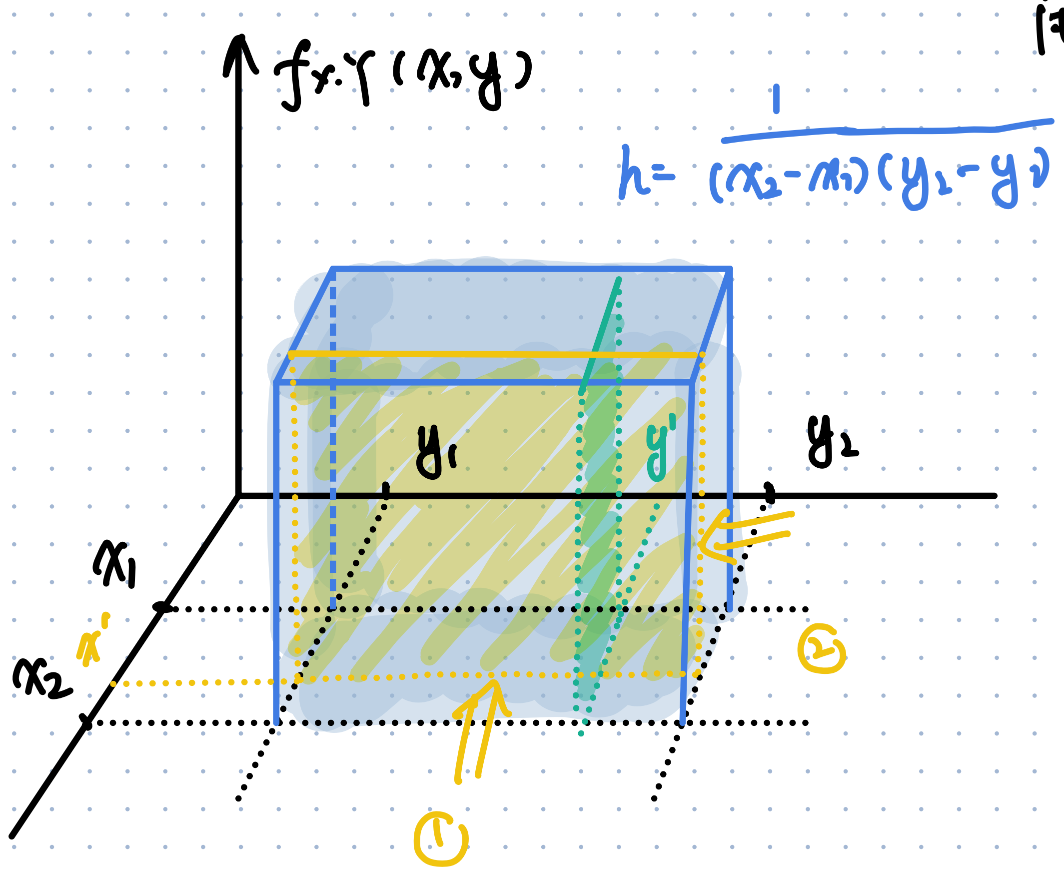


$C = \{[x_3, x_4] \times [y_3, y_4]\}$

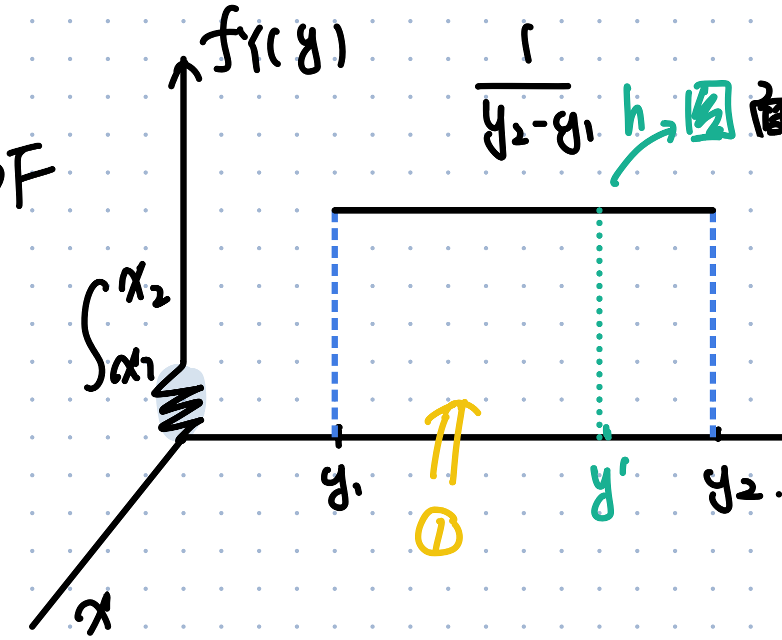
$P((X,Y) \in C) = \int_{y_3}^{y_4} \int_{x_3}^{x_4} f_{X,Y}(x,y)\, dx\, dy =$ 

周辺PDF

$h = \dfrac{1}{(x_2-x_1)(y_2-y_1)}$　Yの周辺PDF

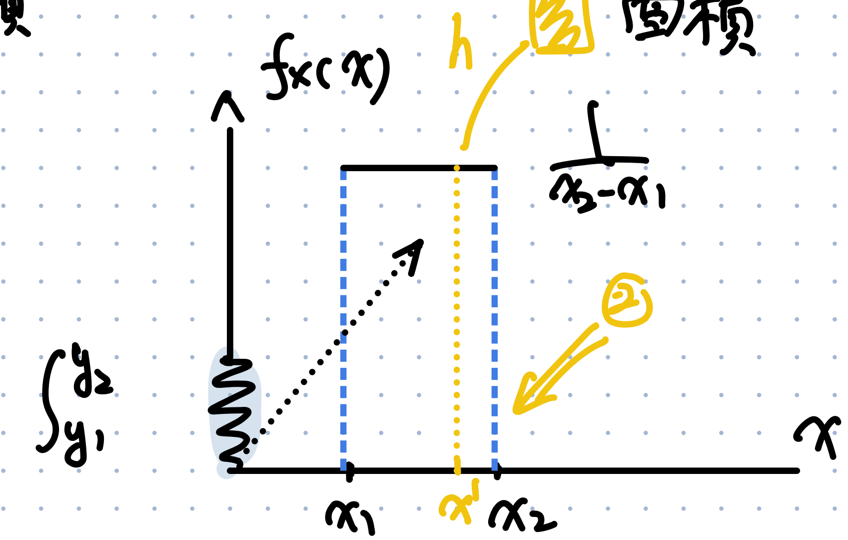


変形 →

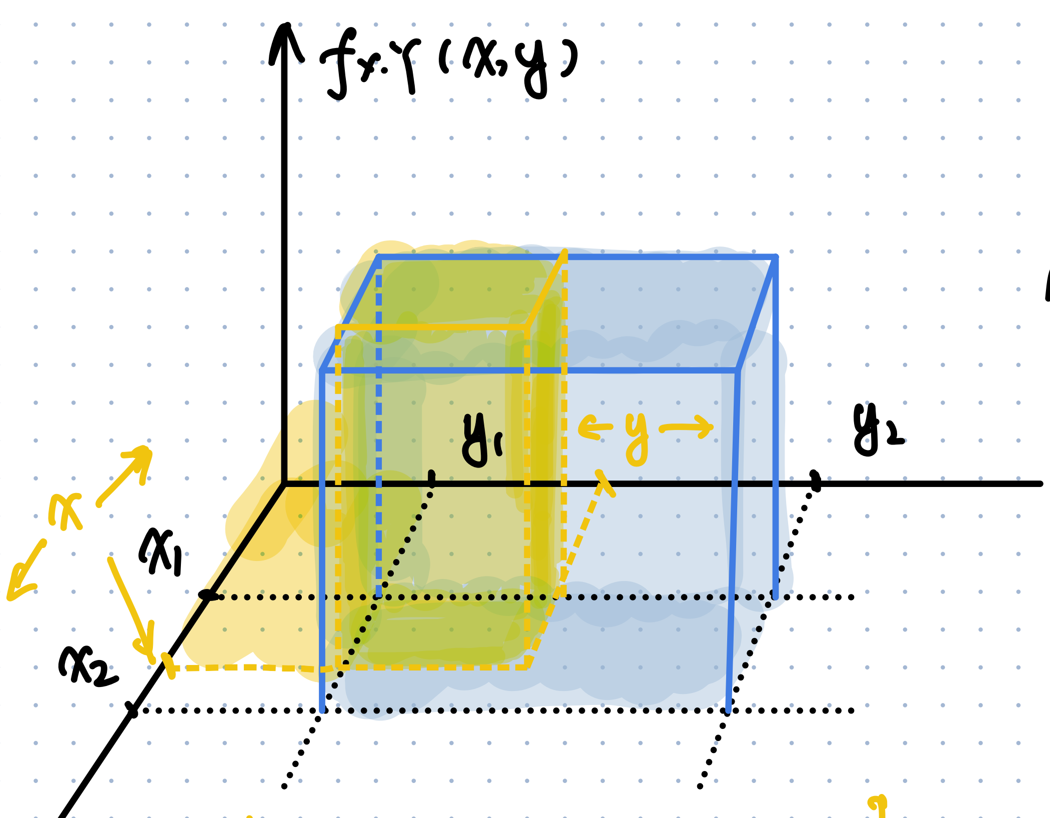


$$\int_{x_1}^{x_2} f_{X,Y}(x,y)\,dx = f_Y(y)$$

Xの周辺PDF



$$\int_{y_1}^{y_2} f_{X,Y}(x,y)\,dy = f_X(x)$$



$C' = \{(-\infty, x] \times (-\infty, y]\}$

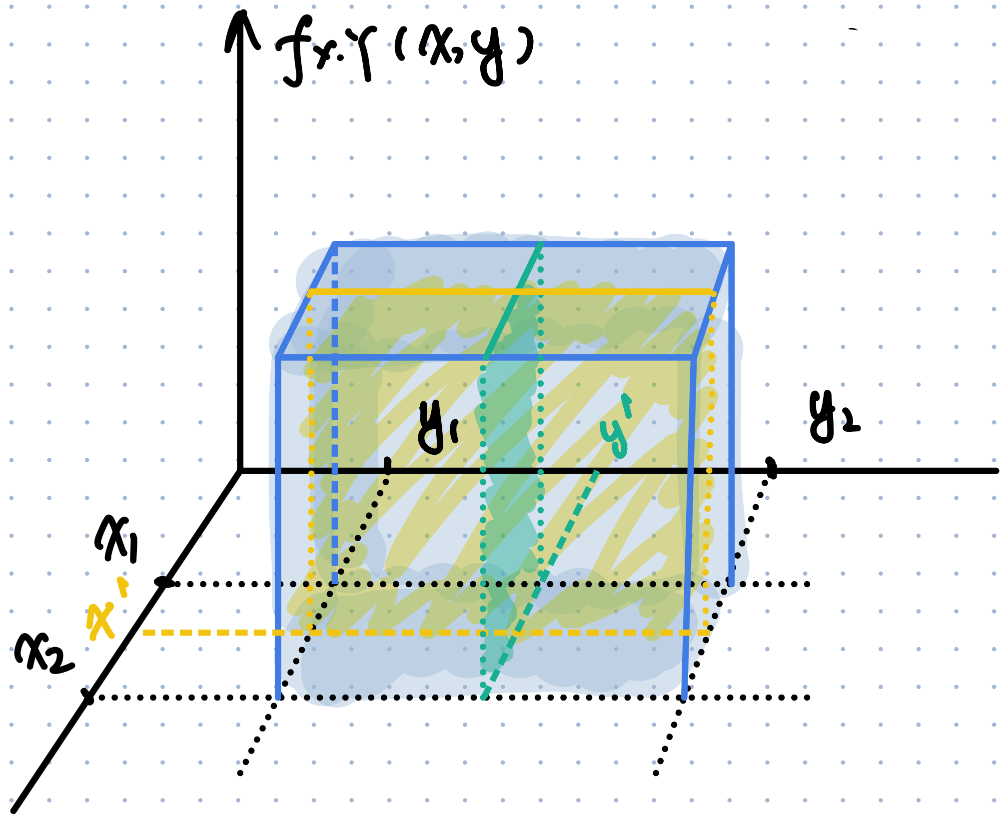
分布関数　$F_{X,Y} = P(X \le x, Y \le y) = P((X,Y) \in C') =$ 

$$= \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(s,t)\,dt\,ds$$

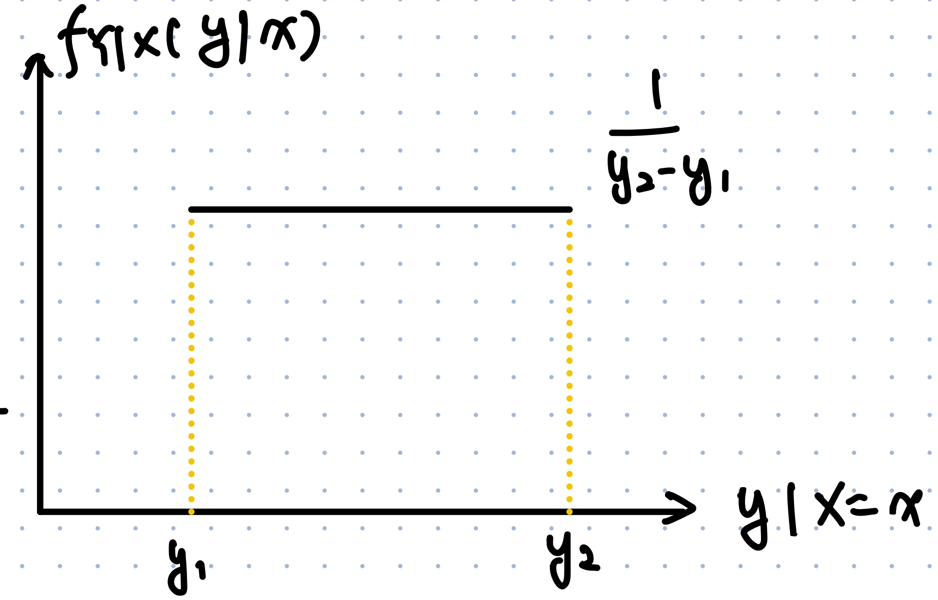
$$= \underbrace{\int_{x_1}^{x}\int_{y_1}^{y}}_{合計範囲} \underbrace{f_{X,Y}(s,t)}_{高} \underbrace{dt\,ds}_{d\sigma}$$

$d\sigma$（微小底面積）$\downarrow 0$　$\neq 0$

条件付き PDF



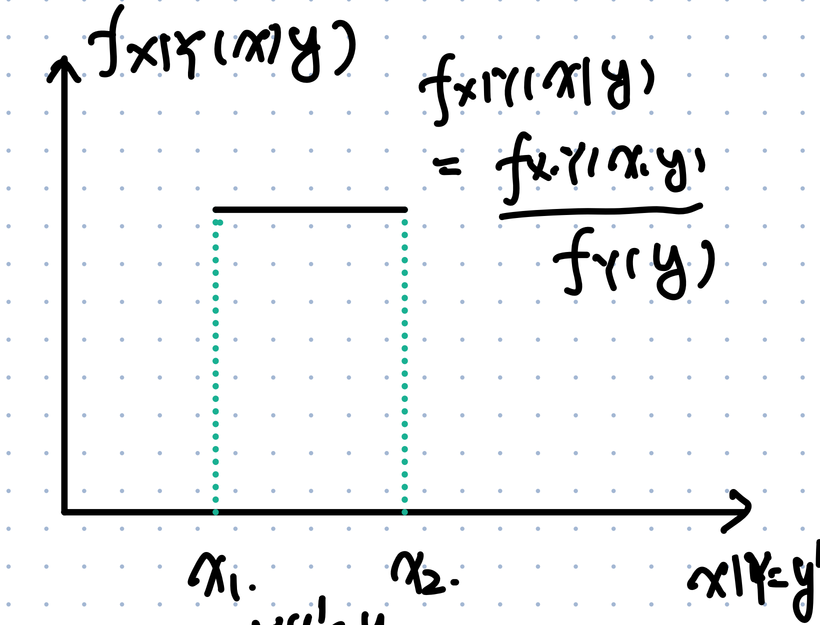
$\longrightarrow \dfrac{\text{同時PDF}}{\text{周辺PDF}}$



$\forall x' \in x$

$$f_{Y|X}(y|x') = \frac{f_{X,Y}(x',y)}{f_X(x')}$$

$$= \frac{\text{同時PDF} — \text{高}}{X=x'\text{の横断面積}}$$



$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$\forall y' \in y$

$$f_{X|Y}(x|y') = \frac{f_{X,Y}(x,y')}{f_Y(y')}$$

$$= \frac{\text{同時PDF} — \text{高}}{Y=y'\text{の横断面積}}$$

確率
関数. $f_X(x_0)$ vs. $P(X=x_0)$

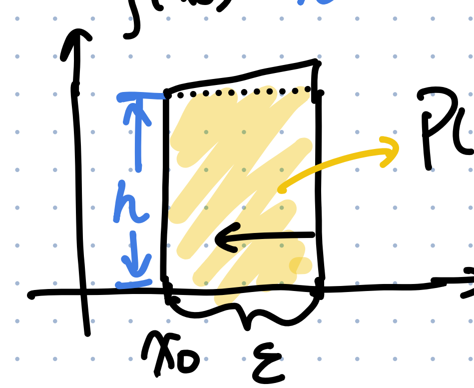
For Discrect R.V : $P(X=x_0) = f_X(x_0)$ PMF

For Continuous R.V: $P(X=x_0)=0 \neq f(x_0)$ ← 関数の高さ $(x=x_0)$ PDF

$$P(x_0 < X < x_0+\varepsilon) = \int_{x_0}^{x_0+\varepsilon} f(x)\,dx \approx f(x_0)\varepsilon$$

↓
0.

$f(x_0) = h$



$P(x_0 < X < x_0+\varepsilon) = S$

↓ $\varepsilon \to 0$ $\neq 0$

$\approx f(x_0)\cdot\varepsilon$

$S \approx h\cdot\varepsilon$

Thm. 4.7. ④

$E[X|Y] = \underline{\int_R x \cdot f_{X|Y}(x|y)\,dx}$ → Yの関数になる

$E\{E[X|Y]\} = \int_R \underline{\quad\quad} f_Y(y)\,dy = \int_R\int_R x \cdot f_{X|Y}(x|y) f_Y(y)\,dx\,dy$

$= \int_R\int_R x f_{X,Y}(x,y)\,dx\,dy = \int_R x \underbrace{\int_R f_{X,Y}(x,y)\,dy}_{f_X(x)}\,dx = \int_R x \cdot f_X(x)\,dx = E[X]$

$Var[X] = E[(X-E[X])^2] = E[(X - E[X|Y] + E[X|Y] - E[X])^2]$

$= E[\{X-E[X|Y]\}^2 + \{E[X|Y]-E[X]\}^2 + 2\{X-E[X|Y]\}\{E[X|Y]-E[X]\}]$

① ② ③

$E[\square] = E[E[\square|Y]]$

① $E[\boxed{\{X-E[X|Y]\}^2}]$

$= E[E\{\boxed{\quad}|Y\}]$

$= E[E\{(X-\mu_{X|Y})^2|Y\}]$

$= E[Var(X|Y)]$

② $E[\{E[X|Y]-E[X]\}^2]$

$= E[\{E[X|Y]-E\{E[X|Y]\}\}^2]$

$= E[(\mu_{X|Y}-E\{\mu_{X|Y}\})^2]$

$= Var[\mu_{X|Y}]$

$= Var\{E[X|Y]\}$

③ $E[\{X-E[X|Y]\}\{E[X|Y]-E[X]\}]$ $E[X] \to \mu_X$

$= E[E[\underline{\quad\quad}|Y]]$

$\cancel{E[\{X-E[X|Y]\}\{E[X|Y]-\mu_X\}|Y]}$

$\cancel{= E[\{X|Y-\mu_X\}\{\mu_X-\mu_X\}]}$

$\cancel{= 0}$

Def 4.8 ⑤

$$E[g(X)h(Y)] = \iint_{R^2} g(x)h(y) f_{XY}(x,y)\,dx\,dy$$

$$= \iint_{R^2} g(x)h(y) f_X(x) f_Y(y)\,dx\,dy$$

$$= \int_R g(x) f_X(x)\,dx \cdot \int_R h(y) f_Y(y)\,dy$$

$$= E[g(X)]\,E[h(Y)]$$

Def. 4.8

$$\mathrm{Var}[aX+bY] = E[(aX+bY)^2] - \{E[aX+bY]\}^2$$

定数

確率変数

$(\Omega, \mathcal{B}, P)$

$$= E[a^2X^2 + b^2Y^2 + 2abXY] - \{E[aX] + E[bY]\}^2$$

$$= E[a^2X^2] + E[b^2Y^2] + E[2abXY] - \{aE[X] + bE[Y]\}^2$$

$$= a^2E[X^2] + b^2E[Y^2] + 2abE[XY] - \{a^2\{E[X]\}^2 + b^2\{E[Y]\}^2 + 2abE[X]E[Y]\}$$

$$= a^2\mathrm{Var}[X] + b^2\mathrm{Var}[Y] + 2ab\,\mathrm{Cov}[X,Y]$$

pp. 15.

(X.Y)の同時分布 $\begin{pmatrix}X\\Y\end{pmatrix} \sim N_2(\mu, \Sigma) = N_2\left(\begin{pmatrix}\mu_1\\\mu_2\end{pmatrix}, \begin{pmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2\\ \rho\sigma_1\sigma_2 & \sigma_2^2\end{pmatrix}\right)$ $k=2.$

$$|\Sigma| = \sigma_1^2\sigma_2^2 - 2\rho^2\sigma_1^2\sigma_2^2 = \sigma_1^2\sigma_2^2(1-\rho^2)$$

$$\begin{bmatrix}A & B\\ C & D\end{bmatrix}^{-1} = \frac{1}{AD-BC}\begin{bmatrix}D & -B\\ -C & A\end{bmatrix}$$

$$|\Sigma|^{\frac{1}{2}} = \sigma_1\sigma_2\sqrt{1-\rho^2}$$

$$\begin{vmatrix}A & B\\ C & D\end{vmatrix} = AD-BC$$

$$(x-\mu) = \begin{pmatrix}x_1-\mu_1\\ y-\mu_2\end{pmatrix} \qquad (x-\mu)^T = (x_1-\mu_1 \ . \ y-\mu_2)$$

$$\Sigma^{-1} = \frac{1}{|\Sigma|}\begin{pmatrix}\sigma_2^2 & -\rho\sigma_1\sigma_2\\ -\rho\sigma_1\sigma_2 & \sigma_2^2\end{pmatrix} = \begin{pmatrix}\frac{\cancel{\sigma_2^2}}{\sigma_1^2\cancel{\sigma_2^2}(1-\rho^2)} & -\frac{\rho\cancel{\sigma_1\sigma_2}}{\sigma_1^2\sigma_2^2(1-\rho^2)}\\ -\frac{\rho\cancel{\sigma_1\sigma_2}}{\sigma_1^2\sigma_2^2(1-\rho^2)} & \frac{\cancel{\sigma_1^2}}{\cancel{\sigma_1^2}\sigma_2^2(1-\rho^2)}\end{pmatrix} = \frac{1}{1-\rho^2}\begin{pmatrix}\frac{1}{\sigma_1^2} & \frac{\rho}{\sigma_1\sigma_2}\\ \frac{\rho}{\sigma_1\sigma_2} & \frac{1}{\sigma_2^2}\end{pmatrix}$$

$$\underbrace{\underbrace{(x-\mu_1 \ . \ y-\mu_2)}_{1\times2}\underbrace{\begin{pmatrix}\frac{1}{\sigma_1^2} & \frac{\rho}{\sigma_1\sigma_2}\\ \frac{\rho}{\sigma_1\sigma_2} & \frac{1}{\sigma_2^2}\end{pmatrix}}_{2\times2}}_{1\times2}\underbrace{\begin{pmatrix}x-\mu_1\\ y-\mu_2\end{pmatrix}}_{2\times1} = \underbrace{\left(\frac{(x-\mu_1)}{\sigma_1^2} + \frac{\rho(y-\mu_2)}{\sigma_1\sigma_2} , \frac{\rho(x-\mu_1)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)}{\sigma_2^2}\right)}_{1\times2}\underbrace{\begin{pmatrix}x-\mu_1\\ y-\mu_2\end{pmatrix}}_{2\times1}$$

$$= \underbrace{\left\{\frac{(x-\mu_1)^2}{\sigma_1^2} + \frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right\}}_{1\times1}$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left\{-\frac{1}{2}\frac{(x-\mu_1)^2}{\sigma_1^2}\right\}$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{\frac{1}{\cancel{2\pi}\,\cancel{\sigma_1}\sigma_2\sqrt{1-\rho^2}}}{\frac{1}{\cancel{\sqrt{2\pi}}\,\cancel{\sigma_1}}} \cdot \exp\left\{-\frac{1}{2}\frac{1}{1-\rho^2}\left\{\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - \frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right\}\right.$$

$$\left. + \frac{1}{2}\frac{1}{1-\rho^2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2 \cdot (1-\rho^2)\right\}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2}\cdot\frac{1}{1-\rho^2}\cdot\left\{\rho^2\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\frac{(x-\mu_1)}{\sigma_1}\cdot\frac{(y-\mu_2)}{\sigma_2} + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right\}\right\}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2}\cdot\frac{1}{1-\rho^2}\left\{\left(\frac{y-\mu_2}{\sigma_2}\right)^2 - \frac{2\rho(x-\mu_1)(y-\mu_2)\cdot\sigma_2}{\sigma_1\,\sigma_2^2} + \frac{(x-\mu_1)^2}{\sigma_1^2}\rho^2\right\}\right\}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2}\cdot\frac{1}{1-\rho^2}\cdot\frac{1}{\sigma_2^2}\left\{\underbrace{(y-\mu_2)^2 - 2\frac{\rho\sigma_2}{\sigma_1}(x-\mu_1)(y-\mu_2) + \frac{(x-\mu_1)^2}{\sigma_1^2}\rho^2\cdot\sigma_2^2}_{\left\{(y-\mu_2) - \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)\right\}^2}\right\}\right\}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \cdot \exp\left\{-\frac{1}{2}\cdot\frac{1}{\sigma^2}\cdot\{y-\mu\}^2\right\}$$

$\sigma = \sigma_2\sqrt{1-\rho^2}$

$\mu = \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)$

Thm 4.7 — ③

① $E[X] = \mu_X$ 定数 ② $E[X] = E\{E[X|Y]\}$ ③ $E[X|Y] = E\{E[X|Y]|Y\}$

$$E\{(X - E[X|Y])(E[X|Y] - E[X])\}$$

$$= E\left[E\{(\quad)(\quad)|Y\} \right] \quad ②$$

$$= E\left[E\{(\overset{A}{X \cdot E[X|Y]} - \overset{B}{XE[X]} - \overset{C}{E[X|Y]E[X|Y]} + \overset{D}{E[X]E[X|Y]})|Y\} \right]$$

$$= E\left[\overset{A}{E\{XE[X|Y]|Y\}} - \overset{B}{E\{X\mu_X|Y\}} - \overset{C}{E\{E[X|Y]E[X|Y]|Y\}} + \overset{D}{E\{\mu_X E[X|Y]|Y\}} \right]$$

$$= E\left[\underline{E[X|Y] \cdot E[X|Y]} - \underline{\mu_X E[X|Y]} - \underline{E[X|Y]E[X|Y]} + \underline{\mu_X E[X|Y]} \right]$$

$$= 0$$

$$= \int_x E[X|Y] f_{X|Y}(x|y)\,dx$$

$$= \int_x \cdot \underline{\int_x x f_{X|Y}(x|y)\,dx} \cdot f_{X|Y}(x|y)\,dx$$

$$= \int_x f(y) f_{X|Y}(x|y)\,dx$$

$$= f(y) \cdot \underbrace{\int_x f_{X|Y}(x|y)\,dx}_{=1}$$

$$= f(y)$$

$$A:\ E\{X E[X|Y]|Y\}$$

$$\overset{\text{Def}}{=} \int_x x \cdot E[X|Y] \cdot f_{X|Y}(x|y)\,dx$$

$$= \int_x x \cdot \int_x x \cdot f_{X|Y}(x|y)\,dx \cdot f_{X|Y}(x|y)\,dx$$

$$= \left[\int_x x \cdot f_{X|Y}(x|y)\,dx\right]^2 = \{E[X|Y]\}^2$$

#5.

Thm. 5.6

R.V. defined upon $(\Omega, \mathcal{B}, P)$

$$M_{X+Y}(t) = E\left[e^{(X+Y)t}\right]$$

$$= E\left[e^{Xt} \cdot e^{Yt}\right]$$

$(X \perp Y \Rightarrow E[XY] = E[X]E[Y])$

$$\overset{X \perp Y}{=} E[e^{tX}] \cdot E[e^{tY}]$$

$$= M_X(t) \cdot M_Y(\cdot)$$

$\downarrow$ $X_1 \cdots X_n$

Thm. 5.8

Ex. 5.7

$$M_X(t) = \exp\left\{\mu_1 t + \frac{\sigma_1^2 t^2}{2}\right\}$$

$$M_Y(t) = \exp\left\{\mu_2 t + \frac{\sigma_2^2 t}{2}\right\}$$

$X \perp Y.$

$Z = X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

$$M_Z(t) = \exp\left\{(\mu_1 + \mu_2)t + \frac{\sigma_1^2 + \sigma_2^2}{2} \cdot t^2\right\}$$

$$= \exp\left\{\mu_1 t + \frac{\sigma_1^2}{2} t\right\} \cdot \exp\left\{\mu_2 t + \frac{\sigma_2^2}{2} t\right\}$$

$$= M_X(t) \cdot M_Y(t)$$

Ex. 5.9.

(2)

① $\{X_i\}_{i=1\dots n} \sim \text{Gamma}(\alpha_i, \beta)$

② $X_i \perp X_j \quad (i \neq j)$

MGF of Gamma(α, β): $M(t) = (1-\beta t)^{-\alpha}$

$Z = X_1 + X_2 + \cdots + X_n$

$$M_Z(t) = M_{X_1}(t) \cdot M_{X_2}(t) \cdots M_{X_n}(t)$$
$$= \{(1-\beta t)^{-\alpha_1}\}\{(1-\beta t)^{-\alpha_2}\} \cdots \{(1-\beta t)^{-\alpha_n}\}$$
$$= (1-\beta t)^{-(\alpha_1+\alpha_2+\cdots+\alpha_n)}$$

$\Rightarrow Z \sim \text{Gamma}\left(\sum_{i=1}^{n} \alpha_i, \beta\right)$

$X_i \perp X_j$ $\quad X_i \sim \text{Gamma}(\alpha, \beta)$

independently. identically. distributed

独立. 同分布

If: $X_i \overset{iid}{\sim} \text{Gamma}(\alpha, \beta)$

$\Rightarrow Z \sim \text{Gamma}(n\alpha, \beta)$

2

Thm 5.10

$$M_{ax+b}(t) = E\left[e^{(ax+b)t}\right]$$

$$= E\left[e^{ax\cdot t} e^{bt}\right]$$

$$= e^{bt} E\left[e^{(at)X}\right]$$

$$= e^{bt} \cdot M_X(at)$$

$$\xrightarrow{X_1 .\ X_2 \cdots\ X_n}$$

Thm 5.11

$$Z = (a_1 X_1 + b_1) + (a_2 X_2 + b_2) + \cdots + (a_n X_n + b_n)$$

$$M_Z(t) = E\left[e^{t\{(a_1 x_1 + b_1) + \cdots + (a_n X_n + b_n)\}}\right]$$

$$= E\left[e^{t(b_1 + \cdots b_n)} e^{t a_1 X_1 + \cdots + t a_n X_n}\right]$$

$$= e^{t\Sigma b_i} E\left[e^{t a_1 X_1}\right] \cdots E\left[e^{t a_n X_n}\right]$$

$$= e^{t\Sigma b_i} \cdot M_{X_1}(a_1 t) \cdots M_{X_n}(a_n t)$$

4

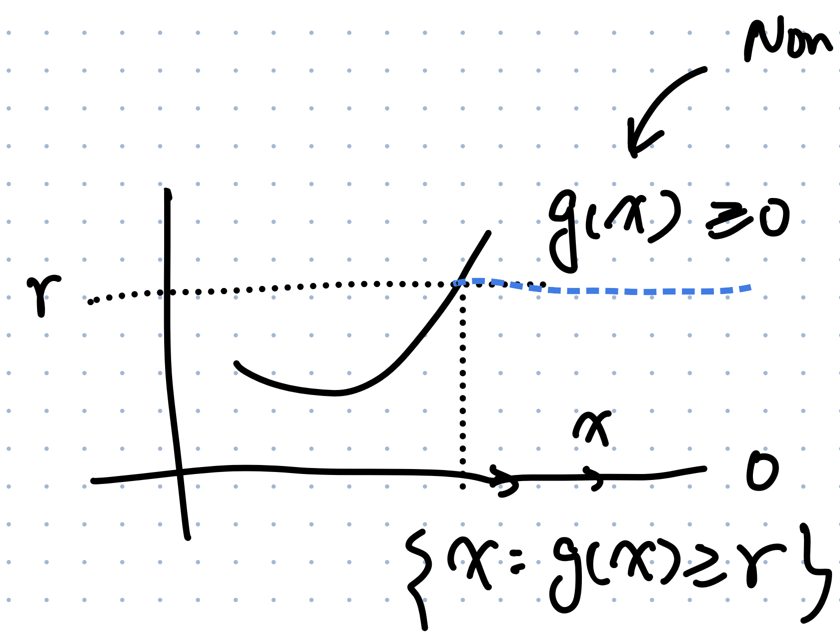
Thm 5.13. Chebychev's inequality.

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f_X(x)\, dx$$

$$\geq \int_{\{x:\, g(x) \geq r\}} g(x) \cdot f_X(x)\, dx$$

$$\geq r \cdot \int_{\{x:\, g(x) \geq r\}} f_X(x)\, dx$$

$$= r \cdot P(g(X) \geq r)$$

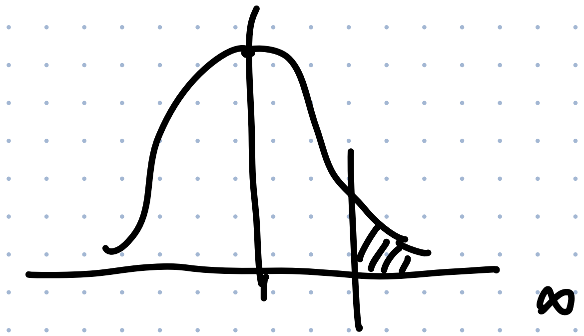


$$\Big\downarrow \; g(X) = X$$

Thm 5.15. Markov's inequality.

Ex. 5.16.

Z:



$t > 0$

$$P(Z \geq t) = \frac{1}{\sqrt{2\pi}} \int_t^{\infty} e^{-\frac{z^2}{2}} dz$$

$$\leq \frac{1}{\sqrt{2\pi}} \int_t^{\infty} \frac{z}{t} \cdot e^{-\frac{z^2}{2}} dz \qquad z \geq t \Rightarrow \frac{z}{t} \geq 1$$

$$= \frac{1}{\sqrt{2\pi}} \cdot -\frac{1}{t} \int_t^{\infty} e^{-\frac{z^2}{2}} d\left(-\frac{z^2}{2}\right)$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \left(-\frac{1}{t}\right) \cdot \left(0 - e^{-\frac{t^2}{2}}\right) = \frac{1}{\sqrt{2\pi}} \cdot \frac{e^{-\frac{t^2}{2}}}{t}$$

$$\rightarrow P(|Z| \geq t) = 2 \times P(Z \geq t)$$

$$P(X \geq a) \overset{t>0}{=} P(tX \geq ta)$$

$$= P(e^{tX} \geq e^{ta}) \leq \frac{E[e^{tX}]}{e^{ta}} \qquad E[e^{tX}] \rightarrow M_X(t)$$

$$\Rightarrow P(X \geq a) \leq e^{-ta} M_X(t)$$

Thm 5.17 ⑥

① $|E[XY]| \leq E[|XY|]$

$Z = XY \quad f_Z(z) = f_{X,Y}(x,y)$ $\quad |\int f| \leq \int |f|$

積分の三角不等式：

$E[Z] = \int_{\mathbb{R}} z \cdot f_Z(z)\,dz$

$|E[Z]| = \left|\int_{\mathbb{R}} z \cdot f_Z(z)\,dz\right| \leq \int_{\mathbb{R}} |z \cdot f_Z(z)|\,dz \overset{f_Z(z) \geq 0}{=} \int_{\mathbb{R}} |z| f_Z(z)\,dz = E[|Z|]$

② $a = \dfrac{|X|}{\{E[|X|^p]\}^{1/p}} \qquad b = \dfrac{|Y|}{\{E[|Y|^q]\}^{1/q}}$ $\qquad \xrightarrow{p=q=2}$ Cauchy-Schwarz 不等式

$E\{\ \} = \frac{1}{p} \qquad E\{\ \} = \frac{1}{q} \Rightarrow \frac{1}{p} + \frac{1}{q} = 1$

According to Supp. 5.19.

$$\frac{1}{p}\left\{\frac{|X|}{(E[|X|^p])^{1/p}}\right\}^p + \frac{1}{q}\left\{\frac{|Y|}{(E[|Y|^q])^{1/q}}\right\}^q = \frac{1}{p}\frac{|X|^p}{E[|X|^p]} + \frac{1}{q}\frac{|Y|^q}{E[|Y|^q]}$$

$$\geq \frac{|XY|}{\{E[|X|^p]\}^{1/p}\{E[|Y|^q]\}^{1/q}} \rightarrow E\{\ \} = \frac{E|XY|}{\boxed{}} \leq 1$$

Thm 5.23.

三角不等式

$|X+Y| \le |X| + |Y|$

$$E[|X+Y|^p] = E[\,|X+Y|\,|X+Y|^{p-1}]$$

①

$$\le E[|X|\,|X+Y|^{p-1}] + E[|Y|\,|X+Y|^{p-1}]$$

② ③

Holder's 不等式

(条件)

$\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow pq = p+q.$

②

$$E[|X|\,|X+Y|^{p-1}] \le E[|X|^p]^{1/p} \cdot E[|X+Y|^{(p-1)q}]^{1/q}$$

$(p-1)q = pq - q = p$

$$= E[|X|^p]^{1/p} \cdot E[|X+Y|^p]^{\frac{1}{q}}$$

$1 - \frac{1}{q} = \frac{1}{p}$

③

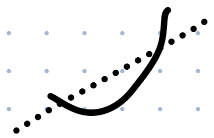
$$E[|Y|\,|X+Y|^{p-1}] \le E[|Y|^p]^{1/p} \cdot E[|X+Y|^p]^{\frac{1}{q}}$$

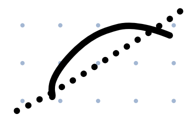
$$\frac{①}{E[|X+Y|^p]^{\frac{1}{q}}} = E[|X+Y|^p]^{1-\frac{1}{q}} \le E[|X|^p]^{1/p} + E[|Y|^p]^{1/p}$$

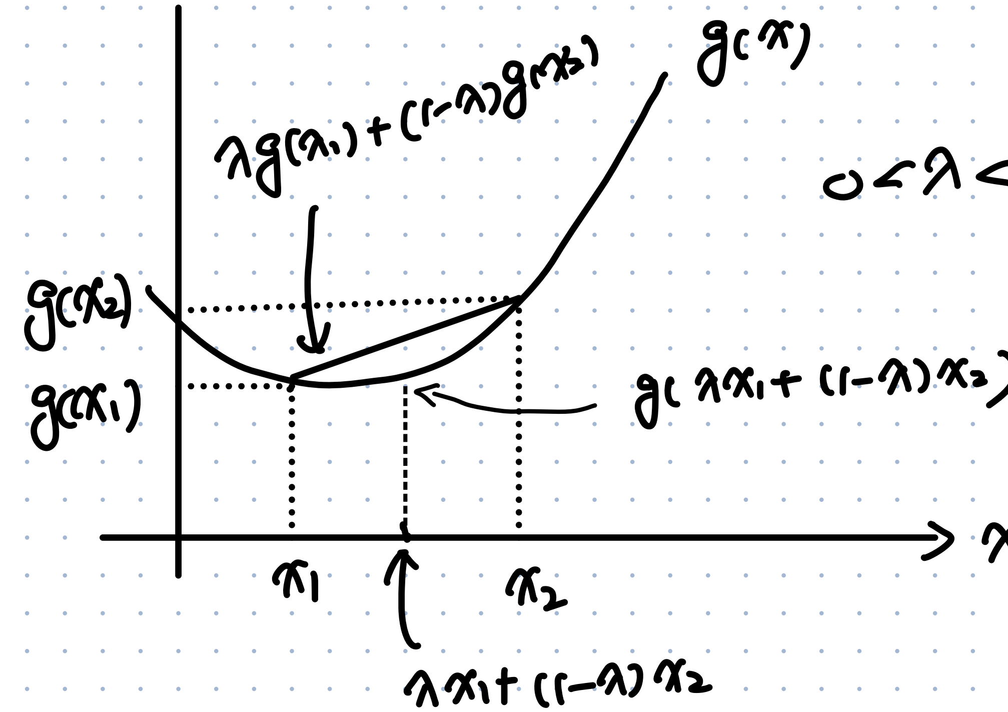
$1 - \frac{1}{q} \to \frac{1}{p}$

8

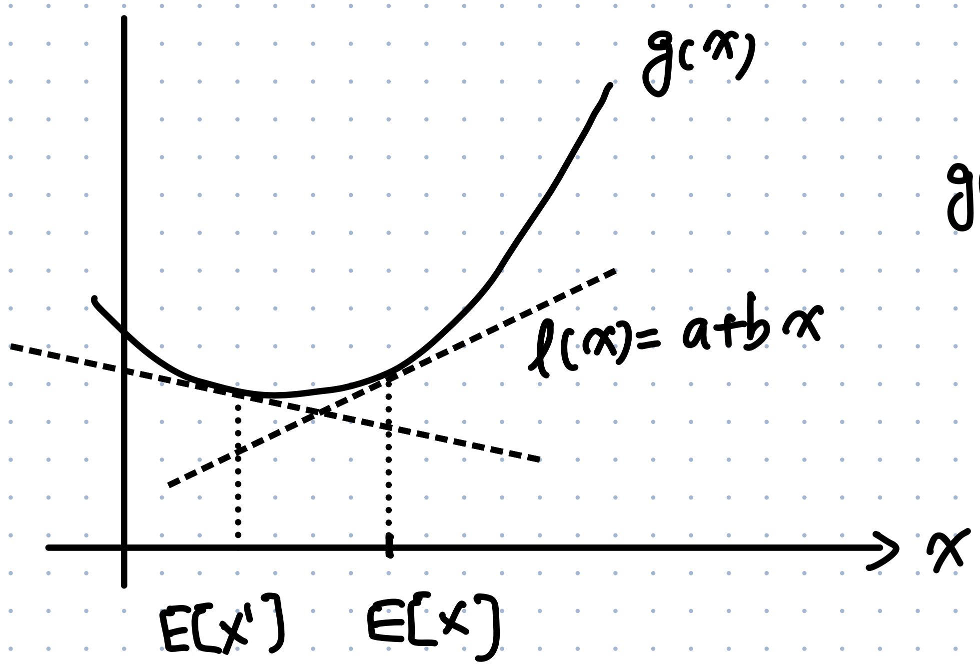
Thm 5.24.

Convex function  凸関数

concave function  凹関数.



$0<\lambda<1$



$g(x) \geq l(x)$

$$
\begin{aligned}
E[g(X)] &\geq E[a+bX] \\
&= a+bE[X] \\
&= l(E[X]) \\
&= g(E[X])
\end{aligned}
$$

$x = E[X] = \mu \quad l(\mu) = g(\mu)$

#6.

Ex 6.5: (離散變數)

$X \sim Bin(n, p)$ $\quad f_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$

$g(x) = n - x$

$Y = N - X \Rightarrow X = N - Y$

$$f_Y(y) = \sum_{x \in g^{-1}(x)} f_X(x)$$

$$= f_X(n-y)$$

$$= \binom{n}{n-y} p^{n-y} (1-p)^y$$

$$= \binom{n}{y} (1-p)^y p^{n-y}$$

$$= Bin(y, 1-p)$$

$\Rightarrow Y \sim Bin(y, 1-p)$

Ex 6.6

$X \sim Unif(0,1)$ $\quad f_X(x) = \begin{cases} 1 & x \in (0,1) \\ 0 & \text{otherwise} \end{cases}$

$$F_X(x) = \int_0^x dx = x$$

$Y = g(X) = -\log X \rightarrow$ CDF?

① $\frac{d}{dx} g(x) = -\frac{1}{x} < 0 \quad \forall x \in (0,1)$

$g(x) \searrow$

② $\mathcal{X} = (0,1)$

$\mathcal{Y} = (0, \infty) \quad \forall y > 0, \quad y = -\log x \Rightarrow x = e^{-y}$

$\Rightarrow g^{-1}(y) = e^{-y} \quad \forall y > 0$

③ Thm 6.3 #2.

$F_Y(y) = 1 - F_X(g^{-1}(y)) =$

$= 1 - F_X(e^{-y}) = 1 - e^{-y} \quad \forall y > 0$

$F_Y(y) = 0$. otherwise

Ex 6.7

$X \sim \text{Gamma}(n, \beta)$ $\quad f_X(x) = \begin{cases} \frac{1}{(n-1)!\beta^n} x^{n-1} e^{-\frac{x}{\beta}} & x>0 \\ 0 & \text{otherwise} \end{cases}$

$(n \in \mathbb{Z}^+, \beta > 0)$

$Y = g(X) = \frac{1}{X} \rightarrow$ PDF? (Thm 6.4)

① $x = (0, \infty)$ $\quad y = g(x) = \frac{1}{x}$

$y = (0, \infty)$ $\quad x = \frac{1}{y} = g^{-1}(y)$

② $\left|\frac{d}{dy} g^{-1}(y)\right| = \left|-\frac{1}{y^2}\right| = \frac{1}{y^2} \quad \forall y > 0$

③ $f_Y(y) = f_X(g^{-1}(y)) \left|\frac{d}{dy} g^{-1}(y)\right|$

$$= \frac{1}{(n-1)!\beta^n} \left(\frac{1}{y}\right)^{n-1} e^{-\frac{1}{\beta y}} \cdot \frac{1}{y^2}$$

$$= \frac{1}{(n-1)!\beta^n} \left(\frac{1}{y}\right)^{n+1} e^{-\frac{1}{\beta y}} \quad \forall y > 0$$

$f_Y(y) = 0$. otherwise.

Ex 6.8 $\quad X \sim f_X(x)$

$Y = g(X) = X^2 \rightarrow$ CDF? PDF?

X^2 is not monotonic function.

According to Def. of CDF.

$$F_Y(y) = P(Y \le y)$$

$$= P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$$

$$= P(X \le \sqrt{y}) - P(X \le -\sqrt{y})$$

$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

$$= -\frac{1}{2} y^{-\frac{1}{2}} f_X(\sqrt{y}) - \left(\frac{1}{2} y^{-\frac{1}{2}} f_X(-\sqrt{y})\right)$$

$$= -\frac{1}{2\sqrt{y}} f_X(\sqrt{y}) - \frac{1}{2\sqrt{y}} f_X(-\sqrt{y})$$

$$= -\frac{1}{2\sqrt{y}}\left[\underbrace{f_X(\sqrt{y})}_{x>0} + \underbrace{f_X(-\sqrt{y})}_{x<0}\right] \quad \forall y > 0$$

Ex 6.9.

$X \sim N(0,1) \quad f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad x \in \mathbb{R}$

$Y = g(X) = X^2$

$\forall x<0 \quad g(x) \searrow \quad \rightarrow$

$\forall x>0 \quad g(x) \nearrow \quad \rightarrow \quad y \in (0, +\infty)$

① $A_0 = \{0\}$

$A_1 = (-\infty, 0) \quad g_1(x) = x^2 \quad g_1^{-1}(y) = -\sqrt{y} \quad \forall y>0$

$A_2 = (0, +\infty) \quad g_1(x) = x^2 \quad g_1^{-1}(y) = \sqrt{y} \quad \forall y>0$

② For A_1. $\quad g^{-1}(y) = x = -\sqrt{y}$

$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| = \frac{1}{\sqrt{2\pi}} e^{-(-\sqrt{y})^2/2} \left| -\frac{1}{2\sqrt{y}} \right|$ Ⓐ

For A_2. $\quad g^{-1}(y) = x = \sqrt{y}$

$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| = \frac{1}{\sqrt{2\pi}} e^{-(\sqrt{y})^2/2} \left| \frac{1}{2\sqrt{y}} \right|$ Ⓑ

③ $f_Y(y) =$ Ⓐ + Ⓑ $\quad \forall y>0$

$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{y}} e^{-\frac{y}{2}} = \chi^2(1)$ ← 自由度.

Thm 6.11.

$Y = F_X(X) \quad \forall y \in (0,1) \rightarrow$ CDF.

Def of CDF.

$F_Y(y) = P(Y \le y)$

$= P(F_X(X) \le y)$

$= P(F_X^{-1}(\square) \le F_X^{-1}(\square)) \quad F_X^{-1}(\cdot) \nearrow$

$= P(X \le F_X^{-1}(y))$

CDF of X $= F_X(F_X^{-1}(y))$

$= y \quad \forall 0<y<1$

$=$ unif$(0,1)$

$P(Y \ge 1) = 1 \quad \rightarrow Y \sim$ unif$(0,1)$

$P(Y \le 0) = 0$

#6-2.

1

Ex 6.12

PMF (Def 1.14)

Random Variable (R.V): $X \perp Y$. $X \sim \text{Poisson}(\theta) \to f_X(x) = P(X=x) = \dfrac{\theta^x e^{-\theta}}{x!}$ $x = 0, 1, 2 \dots$

$Y \sim \text{Poisson}(\lambda) \to f_Y(y) = P(Y=y) = \dfrac{\lambda^y e^{-\lambda}}{y!}$ $y = 0, 1, 2 \dots$

① $X \perp Y \Rightarrow \underbrace{f_{X,Y}(x,y)}_{\text{Joint PMF}} \overset{\text{Def 4.8}}{=} f_X(x) \cdot f_Y(y)$

Bivariate R.V. $(X,Y) \to (x,y) \in \mathcal{A} = \{(x,y) : x = 0,1,2\dots \overset{\text{and}}{\wedge} y = 0,1,2\dots\}$.

② Transformation: $\begin{cases} g_1(x,y) = x+y \\ g_2(x,y) = y \end{cases} \xrightarrow{R.V.} \begin{matrix} g_1(X,Y) = X+Y = U \\ g_2(X,Y) = Y = V \end{matrix} \to (u,v) \in \mathcal{B} = \{(u,v) : \underset{= y \in \{0,1,2\dots\}}{v = 0,1,2,\dots} \wedge \underset{= x \in \{0,1,2\dots\}}{u - v = 0,1,2\dots}\}$.

$= \{(u,v) : v = 0,1,2,\dots \wedge u = v, v+1, v+2, \dots\}$

③ $\forall (u,v) \in \mathcal{B}$, $(x,y) \in A_{uv} = \{(x,y) \in \mathcal{A} : x+y = u \wedge y = v\}$

$= \{(u-v, v)\}$

- Joint PMF of (U,V): $f_{U,V}(u,v) = f_{X,Y}(\overset{x}{u-v}, \overset{y}{v}) = \dfrac{\theta^{u-v} e^{-\theta}}{(u-v)!} \cdot \dfrac{\lambda^v e^{-\lambda}}{v!}$ $\begin{matrix} v = 0,1,2,\dots \\ u = v, v+1, v+2, \dots \end{matrix}$

- Marginal PMF of U: $\forall u \geq 0$, $f_{U,V}(u,v) > 0$ s.t. $v = 0, 1, \dots, u$. $(v = u, u-1, u-2, \dots$ and $v \geq 0 \Rightarrow v = 0, 1, \dots, u)$

$(u = 0, 1, 2 \dots)$

$$f_U(u) = \sum_{v=0}^{u} \frac{\theta^{u-v} e^{-\theta}}{(u-v)!} \cdot \frac{\lambda^v e^{-\lambda}}{v!} = e^{-(\theta+\lambda)} \sum_{v=0}^{u} \frac{\theta^{u-v}}{(u-v)!} \frac{\lambda^v}{v!} = \frac{e^{-(\theta+\lambda)}}{u!} \cdot \sum_{v=0}^{u} \frac{\theta^{u-v}}{(u-v)!} \frac{\lambda^v u!}{v!}$$

Binomial Theorem 二項定理.

$$= \sum_{v=0}^{u} \binom{u}{v} \theta^{u-v} \lambda^v = (\theta + \lambda)^u$$

$$= \frac{e^{-(\theta+\lambda)}}{u!} (\theta+\lambda)^u, \quad u = 0, 1, 2, \dots$$

Ex 6.14. ②

$X \perp Y$. $X \sim Beta(\alpha, \beta)$

$Y \sim Beta(\alpha+\beta, \gamma)$

Joint PDF, Def. 2.28. Thm 2.27.

$$\Rightarrow f_{X,Y}(x,y) = f_X(x)\cdot f_Y(y) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \cdot \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha+\beta)\Gamma(\gamma)} y^{\alpha+\beta-1}(1-y)^{\gamma-1} \quad \begin{matrix} 0<x<1 \\ 0<y<1 \end{matrix}$$

① $(x,y) \in A = \{(x,y) : 0<x<1 \wedge 0<y<1\}$

② Transformation:

$$\begin{cases} g_1(x,y) = xy \\ g_2(x,y) = x \end{cases} \xrightarrow{\text{Random Variable}} \begin{matrix} XY = u \\ X = V \end{matrix} \Rightarrow \begin{matrix} X = V \\ Y = \frac{u}{V} \end{matrix} \xrightarrow{\text{variable}} \begin{cases} x = h_1(u,v) = v \\ y = h_2(u,v) = \frac{u}{v} \end{cases}$$

unique solution. $A \leftrightarrow B$.

$(J \neq 0 \Rightarrow g_1 \cdot g_2 : A \leftrightarrow B)$

$(u,v) \in B = \{(u,v) : 0<v<1, 0<\frac{u}{v}<1\} = \{(u,v) : 0<u<v<1\}$

③ Jacobian $J = \det\left(\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}\right) = \det\left(\begin{bmatrix} 0 & 1 \\ \frac{1}{v} & -\frac{u}{v^2} \end{bmatrix}\right) = -\frac{1}{v}$

• Joint PDF of (U,V): $f_{U,V}(u,v) = f_{X,Y}(h_1, h_2)|J| = f_{X,Y}(v, \frac{u}{v})|J| = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} v^{\alpha-1}(1-v)^{\beta-1}\left(\frac{u}{v}\right)^{\alpha+\beta-1}\left(1-\frac{u}{v}\right)^{\gamma-1} \cdot \frac{1}{v} \quad 0<u<v<1$

• Marginal PDF of u: $f_U(u) = \int_u^1 f_{U,V}(u,v)\,dv = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} u^{\alpha-1} \int_u^1 v^{\alpha-1}(1-v)^{\beta-1} u^{\beta} \cdot v^{-(\alpha+\beta)}\left(1-\frac{u}{v}\right)^{\gamma-1} dv$

$$= v^{-1-\beta}(1-v)^{\beta-1} u^{\beta}\left(1-\frac{u}{v}\right)^{\gamma-1}$$

$$= v^{1-\beta} \cdot v^{-2} \cdot (1-v)^{\beta-1} u^{\beta-1} \cdot u \left(1-\frac{u}{v}\right)^{\gamma-1}$$

$$= \left(\frac{u(1-v)}{v}\right)^{\beta-1}\left(1-\frac{u}{v}\right)^{\gamma-1} \cdot \frac{u}{v^2}$$

$$= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} u^{\alpha-1} \int_u^1 \left(\frac{u}{v}-u\right)^{\beta-1}\left(1-\frac{u}{v}\right)^{\gamma-1} \frac{u}{v^2}\,dv$$

$$\begin{cases} t = \dfrac{\frac{u}{v}-u}{1-u} \\ 1-t = \dfrac{1-\frac{u}{v}}{1-u} \end{cases} \quad 0<t<1 \quad \Rightarrow \frac{dt}{dv} = \frac{-\frac{u}{v^2}}{(1-u)} = -\frac{u}{v^2(1-u)}$$

$$\int_0^1 (1-u)^{\beta-1} \cdot t^{\beta-1} (1-u)^{\gamma-1} \cdot (1-t)^{\gamma-1} |u-1|\,dt = (1-u)^{\gamma+\beta-1} \int_0^1 t^{\beta-1}(1-t)^{\gamma-1}\,dt$$

$$= Beta(\beta,\gamma) = \frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\beta+\gamma)}$$

$$= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta+\gamma)} u^{\alpha-1}(1-u)^{\gamma+\beta-1}$$

$0<u<1$

Ex 6.15 ③

$X \sim N(0,1)$ $X \perp Y \Rightarrow f_{X,Y}(x,y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$ $-\infty < x < \infty$, $-\infty < y < \infty$ $(x,y) \in A$

$Y \sim N(0,1)$

① $A = \{(x,y) : -\infty < x < \infty \wedge -\infty < y < \infty\} = \mathbb{R}^2$

unique solution. $A \longleftrightarrow B$

② Transformation $\begin{cases} g_1(x,y) = x+y \\ g_2(x,y) = x-y \end{cases} \xrightarrow{R.V.} \begin{matrix} X+Y = U \\ X-Y = V \end{matrix} \Rightarrow \begin{matrix} X = \frac{U+V}{2} \\ Y = \frac{U-V}{2} \end{matrix} \xrightarrow{\text{variables}} \begin{cases} x = h_1(u,v) = \frac{u+v}{2} \\ y = h_2(u,v) = \frac{u-v}{2} \end{cases}$ $(u,v) \in B$.

$B = \{(u,v) ; -\infty < u < \infty \wedge -\infty < v < \infty\} = \mathbb{R}^2$

③ Jacobian: $J = \det\left(\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}\right) = \det\left(\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}\right) = -\frac{1}{2}$

- Joint PDF: $f_{U,V}(u,v) = f_{X,Y}(h_1, h_2)|J| = f_{X,Y}\left(\frac{u+v}{2}, \frac{u-v}{2}\right)\frac{1}{2} = \frac{1}{2\pi} e^{-\left(\frac{u+v}{2}\right)^2/2} e^{-\left(\frac{u-v}{2}\right)^2/2} \cdot \frac{1}{2} = \frac{1}{4\pi} e^{-\frac{u^2+v^2}{4}}$

$$= \left(\frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2}} \cdot e^{-\frac{u^2}{4}}\right)\left(\frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2}} e^{-\frac{v^2}{4}}\right) \quad -\infty < u < \infty, \; -\infty < v < \infty$$

$$= f_U(u) \cdot f_V(v) \overset{\text{Def 4.8}}{\Rightarrow} U \perp V$$

- Marignal PDF: $f_U(u) = \frac{1}{\sqrt{2\pi}} \cdot \underset{\uparrow \sigma}{\frac{1}{\sqrt{2}}} e^{-\frac{1}{2}\left(\frac{u}{\underset{\uparrow \sigma}{\sqrt{2}}}\right)^2} \Rightarrow U \sim N(0,2)$ $V \sim N(0,2)$

$X+Y \sim N(0,2)$ $X-Y \sim N(0,2)$

Thm 6.16.

U・V. を continuous R.V. とする.

$\forall u \in \mathbb{R}, v \in \mathbb{R}$, $A_u = \{x : g(x) \le u\}$

$B_v = \{y : h(y) \le v\}$

Joint CDF of (U, V) : $F_{U,V}(u,v) = P(U \le u, V \le v)$

$= P(X \in A_u, Y \in B_v)$ ← U・V の definition pp20. pp5.

$= P(X \in A_u) P(Y \in B_v)$ ← $X \perp Y$.

Joint PDF of (U, V) : $f_{U,V}(u,v) = \frac{\partial^2}{\partial u \partial v} F_{U,V}(u,v)$

$= \frac{d}{du} P(X \in A_u) \cdot \frac{d}{dv} P(Y \in B_v)$

$= f_U(u) \cdot f_V(v)$

$\overset{\text{Def 4.8}}{\Rightarrow} U \perp V$

④

Ex. 6.17.

5

① $X \sim N(0,1)$, $Y \sim N(0,1)$

$X \perp Y \Rightarrow f_{X,Y}(x,y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$ $\quad -\infty < x < \infty$, $-\infty < y < \infty$ $\quad (x,y) \in \mathcal{A} = \mathbb{R}^2$

② Transformation : $\begin{cases} g_1(x,y) = \frac{x}{y} \\ g_2(x,y) = |Y| \end{cases} \xrightarrow{R.V.} \begin{matrix} \frac{X}{Y} = U \\ |Y| = V \end{matrix} \Rightarrow \begin{matrix} V = \pm Y \\ U = \pm \frac{X}{Y} \end{matrix} \rightarrow$

$\mathcal{A}$: (x,y), $(-x,-y)$ $\rightarrow \boxed{g_1,\ g_2} \rightarrow$ $\mathcal{B}$: (u,v)

$\mathcal{A} \underset{\not\leftarrow}{\xrightarrow{g_1 g_2}} \mathcal{B}$ $\quad (x \in \mathbb{R})$

$\downarrow$

$A_0 = \{(x,y) : y = 0\}$

$A_1 = \{(x,y) : y > 0\}$

$A_2 = \{(x,y) : y < 0\}$

A_0: $P((X,Y) \in A_0) = P(Y=0) = 0$, $\forall$ continuous Y.

A_1: $\begin{matrix} V = Y \\ U = \frac{X}{Y} \end{matrix} \Rightarrow \begin{matrix} X = UV \\ Y = V \end{matrix} \xrightarrow{\text{Variable}} \begin{cases} x = h_{11}(u,v) = uv \\ y = h_{12}(u,v) = v \end{cases}$ $\quad \mathcal{B} = \{(u,v) : v > 0\}$ $(u \in \mathbb{R})$

A_2: $\begin{matrix} V = -Y \\ U = -\frac{X}{Y} \end{matrix} \Rightarrow \begin{matrix} X = -UV \\ Y = -V \end{matrix} \xrightarrow{\text{variables}} \begin{cases} x = h_{21}(u,v) = -uv \\ y = h_{22}(u,v) = -v \end{cases}$ $\quad \mathcal{B} = \{(u,v) : v > 0\}$ $(u \in \mathbb{R})$

③ Jacobian :

A_1, A_2: $J = \det\left(\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}\right) = \det\left(\begin{bmatrix} \pm v & \pm u \\ 0 & \pm 1 \end{bmatrix}\right) \Rightarrow |J| = |v|$

- Joint PDF: $f_{UV}(u,v) = f_{X,Y}(h_{11}, h_{12})\, v + f_{X,Y}(h_{21}, h_{22})\, |v| = \frac{v}{2\pi} e^{-\frac{(uv)^2}{2}} e^{-\frac{v^2}{2}} + \frac{v}{2\pi} e^{-\frac{(-uv)^2}{2}} e^{-\frac{v^2}{2}} = \frac{v}{\pi} e^{-\frac{u^2v^2+v^2}{2}}$ $\quad -\infty < u < \infty$, $v > 0$

- Marginal PDF: $f_U(u) = \int_0^\infty \frac{v}{\pi} e^{-\frac{v^2(u^2+1)}{2}} dv = \frac{1}{2\pi} \int_0^\infty e^{-\frac{z(u^2+1)}{2}} dz = \frac{1}{2\pi} \int_0^\infty \frac{2}{u^2+1} \cdot \frac{u^2+1}{2} \cdot e^{-\frac{u^2+1}{2} \cdot z} dz$

$z = v^2 > 0$

$dz = 2v\,dv$

λ

$= \frac{1}{2\pi} \cdot \frac{2}{u^2+1} \cdot \int_0^\infty \lambda e^{-\lambda z} dz = \frac{1}{\pi(u^2+1)}$ $\quad -\infty < u < \infty$

Ex 1.14

#7.

Supp 7.6.

1. $\sum_{i=1}^{n}(x_i-a)^2 = \sum_{i=1}^{n}(x_i-\bar{x}+\bar{x}-a)^2$

$= \sum_{i=1}^{n}(x_i-\bar{x})^2 + \underline{2\sum_{i=1}^{n}(x_i-\bar{x})(\bar{x}-a)} + \sum_{i=1}^{n}(\bar{x}-a)^2$

$\hookrightarrow 2(\bar{x}-a)\underbrace{\sum_{i=1}^{n}(x_i-\bar{x})}$

$= \sum_{i=1}^{n}x_i - n\bar{x}$

$= n\bar{x}-n\bar{x}$

$= 0$

$= \underbrace{\sum_{i=1}^{n}(x_i-\bar{x})^2 + \sum_{i=1}^{n}(\bar{x}-a)^2}_{\text{minimized when } a=\bar{x}}$

2. $a=0$.

$\sum_{i=1}^{n}(x_i-0)^2 = \sum_{i=1}^{n}(x_i-\bar{x})^2 + \sum_{i=1}^{n}(\bar{x})^2$

$\downarrow$

$\sum_{i=1}^{n}x_i^2 = \sum_{i=1}^{n}(x_i-\bar{x})^2 + n\bar{x}^2$

$\Rightarrow \sum_{i=1}^{n}(x_i-\bar{x})^2 = \sum_{i=1}^{n}x_i^2 - n\bar{x}^2$

3. $E\{\sum_{i=1}^{n}g(x_i)\}$

$= E\{g(X_1)+g(X_2)+\cdots+g(X_n)\}$

$= E[g(X_1)]+E[g(X_2)]+\cdots+E[g(X_n)]$

$= \sum_{i=1}^{n}E[g(x_i)]$

$= nE[g(X_1)]$

$\because X_i \overset{iid}{\sim} f(x)$

$\therefore E[X_i]=E[X_1] \quad Var[X_i]=Var[X_1]$

4. $Var\{\sum_{i=1}^{n}g(x_i)\}$

$Y=\sum_{i=1}^{n}g(x_i) \Rightarrow Var[Y] = E\{Y-E(Y)\}^2$

$= E\{\sum_{i=1}^{n}g(x_i) - \underbrace{E\{\sum_{i=1}^{n}g(x_i)\}}_{\sum_{i=1}^{n}E[g(x_i)]}\}^2$

$= E\{\sum_{i=1}^{n}(g(x_i)-E[g(x_i)])\}^2$

$= E\left[\{g(X_1)-E[g(X_1)]\}+\cdots+\{g(X_n)-E[g(X_n)]\}\right]^2$

$= E\{g(X_1)-E[g(X_1)]\}^2+\cdots+E\{g(X_n)-E[g(X_n)]\}^2$

$+E(g(X_1)-E[g(X_1)])(g(X_2)-E[g(X_2)])$
$+\cdots$ } $n(n-1)$ 個 $Cov\{g(x_i),g(x_j)\}$ $=0$

$= Var[g(X_1)]+\cdots+Var[g(X_n)] = nVar[g(X_1)]$

Thm 7.7. 统计量/推定量 ②

不偏性 unbiased: $E(\hat{\theta}) = \theta$ ← 真の值.

Supp 7.6-3

1. $E[\bar{X}] = E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n} \underbrace{EX_i}_{\mu} = \frac{1}{n}\cdot n \cdot E[X_1]$

Supp 7.6-4

2. $Var[\bar{X}] = Var\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} Var\left[\frac{1}{n}X_i\right] = \sum_{i=1}^{n}\frac{1}{n^2} Var[X_i] = n\cdot\frac{1}{n^2}\cdot Var[X_1] = n\cdot\frac{1}{n^2}\cdot\sigma^2 = \frac{\sigma^2}{n}$

3. $E[S^2] = E\left[\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2\right]$

Supp 7.6-2

$= E\left[\frac{1}{n-1}\sum_{i=1}^{n} X_i^2 - n\bar{X}^2\right]$

$= \frac{1}{n-1}\left(\sum_{i=1}^{n} EX_i^2 - nE\bar{X}^2\right)$

$= \frac{1}{n-1}\left(\sum_{i=1}^{n} EX_1^2 - nE\bar{X}^2\right)$

$= \frac{1}{n-1}\left(n\cdot(\sigma^2+\mu^2) - n\left(\frac{\sigma^2}{n}+\mu^2\right)\right)$

$= \frac{1}{n-1}(n\sigma^2 - \sigma^2)$

$= \sigma^2$

$Var X_1 = EX_1^2 - (EX_1)^2$

$\sigma^2 = EX_1^2 - \mu^2$

$EX_1^2 = \sigma^2 + \mu^2$

$E\bar{X}^2 = Var[\bar{X}] + (E\bar{X})^2$

$= \frac{\sigma^2}{n} + \mu^2$

Thm 7.8

$\bar{X} = \frac{1}{n}(X_1 + X_2 + \cdots X_n)$.

$M_{\bar{X}}(t) = E[e^{t\bar{X}}]$

$= E\{e^{t\cdot\frac{1}{n}(X_1+\cdots+X_n)}\}$

$= E\{e^{\frac{t}{n}X_1}\,e^{\frac{t}{n}X_2}\cdots e^{\frac{t}{n}X_n}\}$

$= \{E[e^{\frac{t}{n}X}]\}^n$ $\quad M_{X_i}(t) = M_X(t)$

$= \{M_X(\frac{t}{n})\}^n$

Ex 7.9.

$X \sim N(\mu, \sigma^2) \rightarrow M_X(t) = \exp\{\mu t + \frac{\sigma^2 t^2}{2}\}$.

$M_{\bar{X}}(t) = \{M_X(t/n)\}^n$

$= \left[\exp\{\mu\cdot\frac{t}{n} + \frac{\sigma^2\frac{t^2}{n^2}}{2}\}\right]^n$ $\quad (e^a)^b = e^{ab}$

$= \exp\{n\cdot\mu\cdot\frac{t}{n} + n\cdot\sigma^2\cdot\frac{t^2}{n^2}\cdot\frac{1}{2}\}$

$= \exp\{\mu t + \frac{\frac{\sigma^2}{n}t^2}{2}\}$ $\Rightarrow \bar{X} \sim N(\mu, \frac{\sigma^2}{n})$

$X \sim \text{Gamma}(\alpha, \beta) \rightarrow M_X(t) = \frac{1}{(1-\beta t)^\alpha}$

$M_{\bar{X}}(t) = \{M_X(t/n)\}^n$

$= \left\{\frac{1}{(1-\beta\frac{t}{n})^\alpha}\right\}^n$

$= \frac{1}{(1-\beta/n\cdot t)^{\alpha n}}$ $\Rightarrow \bar{X} \sim \text{Gamma}(n\alpha, \frac{\beta}{n})$

Thm 7.10.

2. $\bar{X} \perp S^2$

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2$$

$$= \frac{1}{n-1}\left\{(X_1-\bar{X})^2 + \sum_{i=2}^{n}(X_i-\bar{X})^2\right\}$$

$$\sum_{i=1}^{n}(X_i-\bar{X}) = \sum_{i=1}^{n}X_i - n\bar{X} = 0$$

$$\Rightarrow (X_1-\bar{X}) = -\sum_{i=2}^{n}(X_i-\bar{X})$$

$$= \frac{1}{n-1}\left\{\left[\sum_{i=2}^{n}(X_i-\bar{X})\right]^2 + \sum_{i=2}^{n}(X_i-\bar{X})^2\right\}$$

$$= f(X_2-\bar{X},\ X_3-\bar{X},\ \dots,\ X_n-\bar{X})$$

$\bar{X} \perp S^2 \iff (X_2-\bar{X}, \dots, X_n-\bar{X}) \perp \bar{X}$ (*)

$$\because (X_1 \cdots X_n) \sim f(x_1 \cdots x_n)$$

$$= f(x_1)\cdots f(x_n)$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}}\exp\left\{-\frac{1}{2}\sum_{i=1}^{n}x_i^2\right\}$$

$$-\infty < x_i < \infty$$

Transformation:

$$\begin{array}{lll}
y_1 = \bar{x} & \Rightarrow x_1 = y_1 - (y_2+y_3+\cdots y_n) & \Rightarrow \frac{\partial x_1}{\partial y_1} = 1,\ \frac{\partial x_1}{\partial y_i} = -1\ (i=2\dots n) \\
y_2 = x_2 - \bar{x} & x_2 = y_2 + y_1 & \frac{\partial x_2}{\partial y_1} = 1,\ \frac{\partial x_2}{\partial y_2} = 1,\ \frac{\partial x_2}{\partial y_i} = 0\ (i \neq 1,2) \\
y_3 = x_3 - \bar{x} & x_3 = y_3 + y_1 & \\
\vdots & \vdots & \vdots \\
y_n = x_n - \bar{x} & x_n = y_n + y_1 & \frac{\partial x_n}{\partial y_1} = 1,\ \frac{\partial x_n}{\partial y_n} = 1,\ \frac{\partial x_n}{\partial y_i} = 0\ (i \neq 1,n)
\end{array}$$

$$J = \begin{vmatrix} 1 & -1 & -1 & \cdots & \cdots \\ 1 & 1 & 0 & 0 & 0 \cdots \\ 1 & 0 & 1 & 0 \cdots & \\ \vdots & & & \ddots & \\ 1 & 0 & \cdots & & 1 \end{vmatrix} = \begin{vmatrix} n & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 0 \cdots \\ \vdots & & & \ddots \\ 0 & \cdots & & 1 \end{vmatrix} = n$$

$$f_{Y_1\cdots Y_n}(y_1 \cdots y_n)$$

$$= f_{X_1\cdots X_n}\left(y_1 - \sum_{i=2}^{n}y_i,\ y_2+y_1,\ \cdots,\ y_n+y_1\right)|J|$$

$$= \frac{n}{(2\pi)^{\frac{n}{2}}}\exp\left[-\frac{1}{2}\left\{\left(y_1-\sum_{i=2}^{n}y_i\right)^2 + (y_2+y_1)^2 + \cdots + \cdots (y_n+y_1)^2\right\}\right]$$

$$= \frac{n}{(2\pi)^{\frac{n}{2}}}\exp\left\{-\frac{1}{2}\left(y_1^2 + \left(\sum_{i=2}^{n}y_i\right)^2 + y_2^2+y_1^2 + \cdots + y_n^2+y_1^2 - 2y_1(y_2+\cdots+y_n) + 2y_1y_2 + \cdots + 2y_ny_1\right)\right\}$$

$$= \frac{n}{(2\pi)^{n/2}}\exp\left\{-\frac{1}{2}\left(ny_1^2 + \left(\sum_{i=2}^{n}y_i\right)^2 \overset{=0}{+} \sum_{i=2}^{n}y_i^2\right)\right\}$$

$$= \left[\left(\frac{n}{2\pi}\right)^{\frac{1}{2}}\exp\left\{-\frac{1}{2}ny_1^2\right\}\right]\left[\frac{n^{\frac{1}{2}}}{(2\pi)^{\frac{n-1}{2}}}\exp\left\{-\frac{1}{2}\left[\left(\sum_{i=2}^{n}y_i\right)^2 + \sum_{i=2}^{n}y_i^2\right]\right\}\right]$$

$$= f_{Y_1}(y)\, f_{Y_2\cdots Y_n}(y_1 \cdots y_n) \Rightarrow Y_1 \perp Y_2 \cdots Y_n \iff \bar{X} \perp X_2-\bar{X}, \cdots X_n-\bar{X}$$ (*)

Thm 7.10

3. Induction proof 帰納法.

$\bar{X}_k, S_k^2$ size k の標本の平均と分散. $(X_i \overset{iid}{\sim} N(\mu, \sigma^2))$

$$(n-1)S_n^2 = (n-2)S_{n-1}^2 + \left(\frac{n-1}{n}\right)(X_n - \bar{X}_{n-1})^2$$ (Report)

① $n=1$. $0\ \frac{S_1^2}{\sigma^2} = 0$

② $n=2$. $\frac{1 \times S_2^2}{\sigma^2} = \frac{1}{2\sigma^2}(X_2 - \bar{X}_1)^2 = \frac{1}{2\sigma^2}(X_2 - X_1)^2$

Ex 6.15

$X_2 - X_1 \sim N(0, 2\sigma^2) \Rightarrow \frac{X_2 - X_1}{\sqrt{2}\sigma} \sim N(0,1) \Rightarrow \frac{S_2^2}{\sigma^2} \sim \chi_1^2$

③ $n=k$.

Assume $\frac{(k-1)S_k^2}{\sigma^2} \sim \chi_{k-1}^2$

$n=k+1$.

$$\frac{kS_{k+1}^2}{\sigma^2} = \frac{(k-1)S_k^2}{\sigma^2} + \frac{\left(\frac{k}{k+1}\right)(X_{k+1} - \bar{X}_k)^2}{\sigma^2}$$

$\sim \chi_{k-1}^2 \qquad \to \chi_1^2\ ?? \qquad \to \chi_{k-1}^2 + \chi_1^2$

㊟ $\qquad = \chi_k^2$

㊟ $S_k^2 \perp (X_{k+1} - \bar{X}_k)^2$

$\downarrow$

$\frac{1}{k-1}\sum_{i=1}^{k}(X_i - \bar{X}_k)^2 \qquad X_k \perp X_{k+1} \Rightarrow S_k^2 \perp (X_{k+1} - \bar{X}_k)^2$

$X_{k+1} \sim N(\mu, \sigma^2)$

$\bar{X}_k \sim N\left(\mu, \frac{\sigma^2}{k}\right)$

$X_{k+1} - \bar{X}_k \sim N\left(0, \frac{k+1}{k}\sigma^2\right)$

$$\frac{X_{k+1} - \bar{X}_k}{\sqrt{\left(\frac{k+1}{k}\right)\sigma^2}} \sim N(0,1)$$

$$\frac{(X_{k+1} - \bar{X}_k)^2}{\frac{k+1}{k}\sigma^2} \sim \chi_1^2$$

$$\Rightarrow \frac{kS_{k+1}^2}{\sigma^2} \sim \chi_k^2$$

$$\Rightarrow k = n-1, \quad \frac{(n-1)S_n^2}{\sigma^2} \sim \chi_{n-1}^2$$

Thm 7.11.

$U \sim N(0,1)$

$V \sim \chi^2_p$ $\quad U \perp V. \Rightarrow$ Joint PDF $f_{U,V}(u,v) = f_U(u) f_V(v) = \dfrac{1}{(2\pi)^{\frac{1}{2}}} e^{-\frac{u^2}{2}} \dfrac{1}{\Gamma(\frac{p}{2}) 2^{\frac{p}{2}}} v^{\frac{p}{2}-1} e^{-\frac{v}{2}}$ $\quad -\infty < u < \infty$, $0 < v < \infty$

Transformation: $\begin{cases} t = \dfrac{u}{\sqrt{v/p}} \\ w = v \end{cases} \longrightarrow \begin{cases} u = t\sqrt{\dfrac{w}{p}} \\ v = w \end{cases} \longrightarrow J = \begin{vmatrix} \frac{\partial u}{\partial t} & \frac{\partial u}{\partial w} \\ \frac{\partial v}{\partial t} & \frac{\partial v}{\partial w} \end{vmatrix} = \begin{vmatrix} \sqrt{\frac{w}{p}} & \frac{t}{\sqrt{p}} \cdot \frac{1}{2} w^{-\frac{1}{2}} \\ 0 & 1 \end{vmatrix} = \sqrt{\frac{w}{p}}$

$\Rightarrow f_{T,W}(t,w) = f_{U,V}\left(t\sqrt{\frac{w}{p}}, w\right)\sqrt{\frac{w}{p}}$

$\Rightarrow f_T(t) = \int_0^\infty f_{U,V}\left(t\sqrt{\frac{w}{p}}, w\right)\sqrt{\frac{w}{p}}\, dw$

$$= \frac{1}{(2\pi)^{\frac{1}{2}}} \frac{1}{\Gamma(\frac{p}{2}) 2^{p/2}} \int_0^\infty e^{-\frac{1}{2}\cdot\frac{t^2 w}{p}} w^{\frac{p}{2}-1} e^{-\frac{w}{2}} \left(\frac{w}{p}\right)^{\frac{1}{2}} dw$$

$$= \frac{1}{(2\pi)^{\frac{1}{2}}} \cdot \frac{1}{\Gamma(\frac{p}{2}) 2^{\frac{p}{2}} p^{\frac{1}{2}}} \underbrace{\int_0^\infty e^{-\frac{1}{2}(1+\frac{t^2}{p})w} w^{\frac{p+1}{2}-1} dw}_{= \Gamma(\frac{p+1}{2}) \left[\frac{2}{1+\frac{t^2}{p}}\right]^{\frac{p+1}{2}}}$$

Gamma$\left(\frac{p+1}{2}, 2(1+\frac{t^2}{p})\right)$

8.

①

Thm 8.3.

$$\text{WLLN}: \quad X_1, X_2, \dots \overset{iid}{\sim} (\mu, \sigma^2) \quad \Rightarrow \quad \bar{X}_n \xrightarrow{P} \mu$$

$\sigma^2 < \infty$

Thm 5.13

chebychev's 不等式 $\quad \forall r > 0, \; P(g(X) \geq r) \leq \dfrac{E[g(X)]}{r}$

Thm 7.7-2

$$P(|\bar{X}_n - \mu| \geq \varepsilon) = P((\bar{X}_n - \mu)^2 \geq \varepsilon^2) \leq \frac{E[(\bar{X}_n - \mu)^2]}{\varepsilon^2} = \frac{Var[\bar{X}_n]}{\varepsilon^2} = \frac{\sigma^2}{n\varepsilon^2} \xrightarrow[n\to\infty]{} 0$$

$$\Leftrightarrow P(|\bar{X}_n - \mu| < \varepsilon) = 1 - P(|\bar{X}_n - \mu| \geq \varepsilon) \geq 1 - \frac{\sigma^2}{n\varepsilon^2} \xrightarrow[n\to\infty]{} 1$$

Eg. 8.4.

$$P(|S_n^2 - \sigma^2| \geq \varepsilon) \leq \frac{E[(S_n^2 - \sigma^2)^2]}{\varepsilon^2} = \frac{Var[S_n^2]}{\varepsilon^2} \xrightarrow{n\to\infty} 0 \quad \text{if } Var[S_n^n] \to 0$$

Ex 8.14 Delta method.

Taylor expansion of $g(Y_n)$ around $Y_n = \theta$:

①

剩余项

$$g(Y_n) = g(\theta) + g'(\theta)(Y_n - \theta) + \text{Remainder}$$

$Y_n \to \theta \Rightarrow \text{Remainder} \to 0$

$$g'(\theta)(Y_n - \theta) = g(Y_n) - g(\theta)$$

$$\sqrt{n}\, g'(\theta)(Y_n - \theta) = \sqrt{n}\left[g(Y_n) - g(\theta)\right]$$

$$\because \sqrt{n}(Y_n - \theta) \xrightarrow{D} N(0, \sigma^2)$$

$$\Rightarrow g'(\theta) \cdot \sqrt{n}(Y_n - \theta) \xrightarrow{D} N(0, [g'(\theta)]^2 \sigma^2)$$

②

$$g(Y_n) = g(\theta) + g'(\theta)(Y_n - \theta) + \frac{g''(\theta)}{2!}(Y_n - \theta)^2 + \text{Remainder}$$

$$= g(\theta) + 0 + \frac{g''(\theta)}{2!}(Y_n - \theta)^2 + \text{Remainder}$$

$$g(Y_n) - g(\theta) = \frac{g''(\theta)}{2!}(Y_n - \theta)^2$$

$$\because \left[\frac{\sqrt{n}(Y_n - \theta)}{\sigma}\right]^2 \xrightarrow{D} \chi_1^2$$

$$\frac{n}{\sigma^2}\left(g(Y_n) - g(\theta)\right) = \frac{n}{\sigma^2} \cdot \frac{g''(\theta)}{2!}(Y_n - \theta^2) + \text{Remainder}$$

$$n\left[g(Y_n) - g(\theta)\right] = \sigma^2 \cdot \frac{g''(\theta)}{2!}\left(\frac{\sqrt{n}}{\sigma}(Y_n - \theta)\right)^2 \xrightarrow{D} \sigma^2 \frac{g''(\theta)}{2!}\chi_1^2$$

8.11. CLT.

$$\forall |t|<h,\quad M_{\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma}}(t) \longrightarrow e^{t^2/2} \quad (n\to\infty) \quad \Rightarrow \quad \frac{\sqrt{n}(\bar{X}-\mu)}{\sigma} \dot{\sim} N(0,1)$$

$$Y_i = \frac{X_i-\mu}{\sigma}$$

Y_i の MGF: $M_Y(t)$ 存在する. $\forall |t|<\sigma h$

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} Y_i = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{X_i-\mu}{\sigma} = \frac{1}{\sqrt{n}}\cdot\frac{n\bar{X}_n-n\mu}{\sigma} = \frac{\sqrt{n}(\bar{X}_n-\mu)}{\sigma}$$

$$M_{\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma}}(t) = M_{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}Y_i}(t) \underset{\uparrow\ \text{Thm 5.10}}{=} M_{\sum_{i=1}^{n}Y_i}\left(\frac{t}{\sqrt{n}}\right) \underset{\uparrow\ \text{Thm 5.8}}{=} \left(M_Y\left(\frac{t}{\sqrt{n}}\right)\right)^n \quad \circledast$$

↳ Taylor Expansion する △

$$\triangle\quad M_Y\left(\frac{t}{\sqrt{n}}\right) = E\left[e^{\frac{t}{\sqrt{n}}Y}\right]$$

← supp 1.6 e^x の Taylor expansion

← Thm 1.4

$$= E\left[\sum_{k=0}^{\infty}\frac{\left(\frac{t}{\sqrt{n}}Y\right)^k}{k!}\right] = \sum_{k=0}^{\infty} EY^k\cdot\frac{\left(\frac{t}{\sqrt{n}}\right)^k}{k!} = \sum_{k=0}^{\infty} M_Y^{(k)}(0)\frac{(t/\sqrt{n})^k}{k!}$$

$\rightarrow \forall\left|\frac{t}{\sqrt{n}}\right|<\sigma h \Rightarrow t<\sqrt{n}\sigma h$, $M_Y^{(k)}(0)$ 存在する.

$M_Y^{(0)} = 1 \rightarrow EY^0 = 1$

$M_Y^{(1)} = 0 \rightarrow EY = 0$

$M_Y^{(2)} = 1 \rightarrow EY^2 = Var(Y) - (EY)^2 = 1$

$$= M_Y^{(0)}(0)\frac{(t/\sqrt{n})^0}{0!} + M_Y^{(1)}(0)\cdot\frac{(t/\sqrt{n})}{1!} + M_Y^{(2)}(0)\frac{(t/\sqrt{n})^2}{2!} + \cdots$$

$$= 1 + 0 + \frac{(t/\sqrt{n})^2}{2!} + R_Y\left(\frac{t}{\sqrt{n}}\right)$$

↳ $\sum_{k=3}^{\infty} M_Y^{(k)}(0)\frac{(t/\sqrt{n})^k}{k!}$

$$\lim_{n\to\infty}\frac{R_Y(t/\sqrt{n})}{(t/\sqrt{n})^2} = 0$$

④

$$R_Y\left(\frac{t}{\sqrt{n}}\right) = \sum_{k=3}^{\infty} M_Y^{(k)}(0) \frac{(t/\sqrt{n})^k}{k!}$$

$$\lim_{n\to\infty} \frac{R_Y(t/\sqrt{n})}{(t/\sqrt{n})^2} = 0 \Rightarrow \lim_{n\to\infty} \frac{R_Y(t/\sqrt{n})}{(1/\sqrt{n})^2} = \lim_{n\to\infty} n R_Y(t/\sqrt{n}) = 0$$

$t=0$: $R_Y(0/\sqrt{n}) = 0$

$t\neq 0$: $\lim_{n\to\infty} n \cdot R_Y(t/\sqrt{n}) = 0$

(*) $$\lim_{n\to\infty} \left\{M_Y\left(\frac{t}{\sqrt{n}}\right)\right\}^n = \lim_{n\to\infty}\left[1 + \frac{(t/\sqrt{n})^2}{2!} + R_Y\left(\frac{t}{\sqrt{n}}\right)\right]^n$$

$$= \lim_{n\to\infty}\left[1 + \frac{t^2}{n\cdot 2!} + R_Y\left(\frac{t}{\sqrt{n}}\right)\right]^n$$

$$= \lim_{n\to\infty}\left[1 + \frac{1}{n}\left(\frac{t^2}{2!} + n\cdot R_Y\left(\frac{t}{\sqrt{n}}\right)\right)\right]^n$$

$$= e^{\frac{t^2}{2}}$$ (#) $\leftarrow$ $N(0,1)$ の MGF.

(#) $$\lim_{n\to\infty} a_n = a \Rightarrow \lim_{n\to\infty}\left(1 + \frac{a_n}{n}\right)^n = e^a$$

$$\lim_{n\to\infty} \underbrace{\frac{t^2}{2} + nR_Y\left(\frac{t}{\sqrt{n}}\right)}_{a_n} = \underbrace{\frac{t^2}{2}}_{a.} \Rightarrow \lim_{n\to\infty}\left[1 + \frac{1}{n}(a_n)\right]^n = e^a.$$

#9.

Eg. 9.15.

$$f(y,p) = f(y|p)\cdot f(p) = \left\{ \binom{n}{y} p^{y} (1-p)^{n-y} \right\} \left\{ \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1} (1-p)^{\beta-1} \right\} = \binom{n}{y} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{y+\alpha-1} (1-p)^{n-y+\beta-1}$$

$$f(y) = \int_0^1 f(y,p)\, dp = \int_0^1 \binom{n}{y} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{y+\alpha-1} (1-p)^{n-y+\beta-1}\, dp$$

$$= \binom{n}{y} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \underbrace{p^{y+\alpha-1} (1-p)^{n-y+\beta-1}}_{\parallel \text{ Def 2.28}}\, dp$$

$$B(\alpha', \beta') \frac{1}{B(\alpha', \beta')} p^{\alpha'-1} (1-p)^{\beta'-1}$$

$\alpha' = y+\alpha$

$\beta' = n-y+\beta$

$\alpha'+\beta' = n+\alpha+\beta$

$$= \binom{n}{y} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot B(\alpha', \beta') \cdot \underbrace{\int_0^1 Beta(\alpha', \beta')\, dp}_{=1}$$

$\downarrow$ Thm 2.27

$$= \binom{n}{y} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{\Gamma(\alpha')\Gamma(\beta')}{\Gamma(\alpha'+\beta')}$$

$$= \binom{n}{y} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{\Gamma(y+\alpha)\Gamma(n-y+\beta)}{\Gamma(n+\alpha+\beta)}$$

sample mean

$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$

prior mean.

$E[p]$

$$\hat{p}_B = \left(\frac{n}{\alpha+\beta+n}\right)\left(\frac{y}{n}\right) + \left(\frac{\alpha+\beta}{\alpha+\beta+n}\right)\cdot\left(\frac{\alpha}{\alpha+\beta}\right)$$

Eg. 9.16.

$X_i|\theta \overset{iid}{\sim} N(\theta, \sigma^2)$

$\theta \sim N(\mu, \tau^2)$

$\Rightarrow$

$$f(x_1, \dots x_n | \theta) = \left(\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}\left(\frac{x_i-\theta}{\sigma}\right)^2\right\}\right)^n = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i-\theta}{\sigma}\right)^2\right\} = C \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i-\theta}{\sigma}\right)^2\right\}$$

constant

$$f(\theta) = \frac{1}{\sqrt{2\pi\tau^2}} \exp\left\{-\frac{1}{2}\left(\frac{\theta-\mu}{\tau}\right)^2\right\} = D \exp\left\{-\frac{1}{2}\left(\frac{\theta-\mu}{\tau}\right)^2\right\}$$

constant

$\exp\{-a(\theta-b)^2\}$ $(a>0)$

正規分布の核

① $f(x_1 \dots x_n, \theta) = f(x_1 \dots x_n | \theta) \cdot f(\theta)$

$$= C \cdot D \cdot \exp\left\{-\frac{1}{2}\left[\underline{\sum_{i=1}^{n}\left(\frac{x_i-\theta}{\sigma}\right)^2 + \left(\frac{\theta-\mu}{\tau}\right)^2}\right]\right\}$$

$$= C \cdot D \cdot \exp\left\{-\frac{1}{2}\left[A\left(\theta - \frac{B}{A}\right)^2 - E'\right]\right\}$$

$$= C \cdot D \cdot F \cdot \exp\left\{-\frac{A}{2}\left(\theta - \frac{B}{A}\right)^2\right\}$$

$$= C \cdot D \cdot F \cdot \exp\left\{-\frac{1}{2}\frac{\left(\theta - \frac{B}{A}\right)^2}{\frac{1}{A}}\right\}$$

mean

Var

$$\longrightarrow \sum_{i=1}^{n}\left(\frac{x_i^2 - 2x_i\theta + \theta^2}{\sigma^2}\right) + \frac{\theta^2 - 2\theta\mu + \mu^2}{\tau^2}$$

$$= \frac{\sum_{i=1}^{n} x_i^2 - 2\theta\sum_{i=1}^{n} x_i + n\theta^2}{\sigma^2} + \frac{\theta^2 - 2\theta\mu + \mu^2}{\tau^2}$$

constant

$$= -\frac{2\theta\sum_{i=1}^{n} x_i}{\sigma^2} + \frac{n\theta^2}{\sigma^2} + \frac{\theta^2}{\tau^2} - \frac{2\theta\mu}{\tau^2} + E$$

$$= \theta^2\left(\frac{n}{\sigma^2} + \frac{1}{\tau^2}\right) - 2\theta\left(\frac{\sum_{i=1}^{n} x_i}{\sigma^2} + \frac{\mu}{\tau^2}\right) + E$$

$$= \theta^2 A - 2\theta B + E$$

$$= A\left(\theta^2 - \frac{2\theta B}{A} + \frac{B^2}{A^2}\right) - \frac{B^2}{A} + E$$

$$= A\left(\theta - \frac{B}{A}\right)^2 - E'$$

$A = \left(\frac{n}{\sigma^2} + \frac{1}{\tau^2}\right)$

$B = \left(\frac{\sum_{i=1}^{n} x_i}{\sigma^2} + \frac{\mu}{\tau^2}\right)$

② $$f(x_1 \dots x_n) = \int_{-\infty}^{\infty} C \cdot D \cdot F \cdot \exp\left\{-\frac{1}{2}\frac{\left(\theta - \frac{B}{A}\right)^2}{\frac{1}{A}}\right\} d\theta = \text{constant}$$

③ $$f(\theta | x_1 \dots x_n) = \frac{f(x_1 \dots x_n, \theta)}{f(x_1 \dots x_n)} \propto \exp\left\{-\frac{1}{2}\frac{\left(\theta - \frac{B}{A}\right)^2}{\frac{1}{A}}\right\}$$

$$E[\theta] = \frac{B}{A} = \frac{\frac{\sum_{i=1}^{n} x_i}{\sigma^2} + \frac{\mu}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}} = \frac{\frac{n\bar{x}\tau^2 + \mu\sigma^2}{\sigma^2\tau^2}}{\frac{n\tau^2 + \sigma^2}{\sigma^2\tau^2}} = \frac{n\bar{x}\tau^2 + \mu\sigma^2}{n\tau^2 + \sigma^2} = \frac{n\tau^2}{n\tau^2 + \sigma^2}\bar{x} + \frac{\sigma^2}{n\tau^2 + \sigma^2}\mu$$

$$Var[\theta] = \frac{1}{A} = \frac{\tau^2\sigma^2}{n\tau^2 + \sigma^2}$$

#9.

①

Eg. 9.15.

$$f(y,p) = f(y|p)\cdot f(p) = \left\{\binom{n}{y} p^y (1-p)^{n-y}\right\}\left\{\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1}\right\} = \binom{n}{y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{y+\alpha-1}(1-p)^{n-y+\beta-1}$$

$$f(y) = \int_0^1 f(y,p)\,dp = \int_0^1 \binom{n}{y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{y+\alpha-1}(1-p)^{n-y+\beta-1}\,dp$$

$$= \binom{n}{y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 \underbrace{p^{y+\alpha-1}(1-p)^{n-y+\beta-1}}_{\parallel \text{ Def 2.28}}\,dp$$

$$B(\alpha',\beta')\frac{1}{B(\alpha',\beta')}p^{\alpha'-1}(1-p)^{\beta'-1}$$

$\alpha' = y+\alpha$

$\beta' = n-y+\beta$

$\alpha'+\beta' = n+\alpha+\beta$

$$= \binom{n}{y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\cdot B(\alpha',\beta')\cdot\underbrace{\int_0^1 \text{Beta}(\alpha',\beta')\,dp}_{=1}$$

$\downarrow$ Thm 2.27

$$= \binom{n}{y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\cdot\frac{\Gamma(\alpha')\Gamma(\beta')}{\Gamma(\alpha'+\beta')}$$

$$= \binom{n}{y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\cdot\frac{\Gamma(y+\alpha)\Gamma(n-y+\beta)}{\Gamma(n+\alpha+\beta)}$$

sample mean

$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$

prior mean.

$E[p]$

$$\hat{p}_B = \left(\frac{n}{\alpha+\beta+n}\right)\left(\frac{y}{n}\right) + \left(\frac{\alpha+\beta}{\alpha+\beta+n}\right)\cdot\left(\frac{\alpha}{\alpha+\beta}\right)$$

Eg. 9.16.

$$X_i \mid \theta \overset{iid}{\sim} N(\theta, \sigma^2)$$
$$\theta \sim N(\mu, \tau^2)$$

$\Rightarrow$

$$f(x_1, \dots x_n \mid \theta) = \left(\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}\left(\frac{x_i - \theta}{\sigma}\right)^2\right\}\right)^n = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp\left\{-\frac{1}{2}\sum_{i=1}^n\left(\frac{x_i-\theta}{\sigma}\right)^2\right\} = \underset{\uparrow \text{ constant}}{C} \exp\left\{-\frac{1}{2}\sum_{i=1}^n\left(\frac{x_i-\theta}{\sigma}\right)^2\right\}$$

②

$$f(\theta) = \frac{1}{\sqrt{2\pi\tau^2}} \exp\left\{-\frac{1}{2}\left(\frac{\theta-\mu}{\tau}\right)^2\right\} = \underset{\uparrow \text{ constant}}{D} \exp\left\{-\frac{1}{2}\left(\frac{\theta-\mu}{\tau}\right)^2\right\}$$

$$\exp\{-a(\theta-b)^2\} \quad (a>0)$$

↑ 正規分布の核

① $f(x_1 \dots x_n, \theta) = f(x_1 \dots x_n \mid \theta) \cdot f(\theta)$

$$= C \cdot D \cdot \exp\left\{-\frac{1}{2}\left[\sum_{i=1}^n\left(\frac{x_i-\theta}{\sigma}\right)^2 + \left(\frac{\theta-\mu}{\tau}\right)^2\right]\right\}$$

$$= C \cdot D \cdot \exp\left\{-\frac{1}{2}\left[A\left(\theta - \frac{B}{A}\right)^2 - E'\right]\right\}$$

$$= C \cdot D \cdot F \cdot \exp\left\{-\frac{A}{2}\left(\theta - \frac{B}{A}\right)^2\right\}$$

$$= C \cdot D \cdot F \cdot \exp\left\{-\frac{1}{2}\frac{\left(\theta - \overset{\text{mean}}{\frac{B}{A}}\right)^2}{\underset{\text{Var}}{\frac{1}{A}}}\right\}$$

$$\sum_{i=1}^n\left(\frac{x_i^2 - 2x_i\theta + \theta^2}{\sigma^2}\right) + \frac{\theta^2 - 2\theta\mu + \mu^2}{\tau^2}$$

$$= \frac{\sum_{i=1}^n x_i^2 - 2\theta\sum_{i=1}^n x_i + n\theta^2}{\sigma^2} + \frac{\theta^2 - 2\theta\mu + \mu^2}{\tau^2}$$

$$= -\frac{2\theta\sum_{i=1}^n x_i}{\sigma^2} + \frac{n\theta^2}{\sigma^2} + \frac{\theta^2}{\tau^2} - \frac{2\theta\mu}{\tau^2} + \underset{\text{constant}}{E}$$

$$= \theta^2\left(\frac{n}{\sigma^2} + \frac{1}{\tau^2}\right) - 2\theta\left(\frac{\sum_{i=1}^n x_i}{\sigma^2} + \frac{\mu}{\tau^2}\right) + E$$

$$= \theta^2 A - 2\theta B + E$$

$$A = \left(\frac{n}{\sigma^2} + \frac{1}{\tau^2}\right)$$

$$B = \left(\frac{\sum_{i=1}^n x_i}{\sigma^2} + \frac{\mu}{\tau^2}\right)$$

$$= A\left(\theta^2 - \frac{2\theta B}{A} + \frac{B^2}{A^2}\right) - \frac{B^2}{A} + E$$

$$= A\left(\theta - \frac{B}{A}\right)^2 - E'$$

② $$f(x_1 \dots x_n) = \int_{-\infty}^{\infty} C \cdot D \cdot F \cdot \exp\left\{-\frac{1}{2}\frac{\left(\theta - \frac{B}{A}\right)^2}{\frac{1}{A}}\right\} d\theta = \text{constant}$$

③ $$f(\theta \mid x_1 \dots x_n) = \frac{f(x_1 \dots x_n, \theta)}{f(x_1 \dots x_n)} \propto \exp\left\{-\frac{1}{2}\frac{\left(\theta - \frac{B}{A}\right)^2}{\frac{1}{A}}\right\}$$

$$E[\theta] = \frac{B}{A} = \frac{\frac{\sum_{i=1}^n x_i}{\sigma^2} + \frac{\mu}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}} = \frac{\frac{n\bar{x}\tau^2 + \mu\sigma^2}{\sigma^2\tau^2}}{\frac{n\tau^2 + \sigma^2}{\sigma^2\tau^2}} = \frac{n\bar{x}\tau^2 + \mu\sigma^2}{n\tau^2 + \sigma^2} = \frac{n\tau^2}{n\tau^2 + \sigma^2}\bar{x} + \frac{\sigma^2}{n\tau^2 + \sigma^2}\mu$$

$$\mathrm{Var}[\theta] = \frac{1}{A} = \frac{\tau^2\sigma^2}{n\tau^2 + \sigma^2}$$

Def. 9.18

③

$$E_\theta[(W-\theta)^2] = E_\theta[(W-E[W]+E[W]-\theta)^2] = E_\theta\{(W-E[W])^2 + \underbrace{2(W-E[W])(E(W)-\theta)} + (E[W]-\theta)^2\}$$

$$E_\theta\{W \cdot E(W) - W\theta - (E[W])^2 - \theta E(W)\}$$

$$= E_\theta(W)E(W) - E_\theta(W)\theta - \{E[W]\}^2 - \theta E_\theta(W)$$

$$= 0$$

$$= E_\theta\{(W-E[W])^2\} + (E[W]-\theta)^2$$

$$= \mathrm{Var}_\theta(W) + \{\mathrm{Bias}_\theta(W)\}^2$$

given θ given θ

$E_\theta(W)$: θに依存する

W : θの推定量

$= \int w f(w|\theta)\, dw$

PDF/PMFはθに依存する

$E(W)$: θは必ずしも必要ではない

$= \int w f(w|??)\, dw$

Thm 9.23. (Thm 5.21) ④

Cauchy-Schwarz 不等式: $\{Cov[X,Y]\}^2 \le Var[X]\,Var[Y]$ $\Rightarrow$ ① $Var[X] \ge \dfrac{\{Cov[X,Y]\}^2}{Var[Y]}$ ← Lower Bound of $Var[X]$

② $X \Rightarrow W(X)$

$Y \Rightarrow \frac{\partial}{\partial\theta}\log f(x|\theta)$

推定量 $W(X_1 \cdots X_n)$

$$\frac{d}{d\theta}E_\theta\{W(X)\} = \frac{\partial}{\partial\theta}\int_\chi W(x) f(x|\theta)\,dx \quad \text{(期待値のDef)}$$

$$= \int_\chi W(x)\frac{\partial}{\partial\theta}f(x|\theta)\,dx \quad \text{(条件: 微分可能性)}$$

$$= \int_\chi W(x)\frac{f(x|\theta)}{f(x|\theta)}\frac{\partial}{\partial\theta}f(x|\theta)\,dx$$

$$= \int_\chi \boxed{}\, f(x|\theta)\,dx$$

$$= E\left[W(X)\cdot\frac{\frac{\partial}{\partial\theta}f(X|\theta)}{f(X|\theta)}\right] \quad \text{(期待値のDef)}$$

$$= E\left[W(X)\frac{\partial}{\partial\theta}\log(f(X|\theta))\right] \qquad \frac{\partial}{\partial\theta}\log(F) = \frac{\frac{\partial}{\partial\theta}F}{F}$$

$$\overset{*}{=} Cov\left[W(X), \frac{\partial}{\partial\theta}\log f(X|\theta)\right]$$

Δ: $E\left\{\frac{\partial}{\partial\theta}\log f(X|\theta)\right\}$

$$= \int_\chi \frac{\partial}{\partial\theta}\log f(x|\theta)\cdot f(x|\theta)\,dx \quad \text{(期待値のDef)}$$

$$= \int_\chi \frac{\frac{\partial}{\partial\theta}f(x|\theta)}{f(x|\theta)} f(x|\theta)\,dx \quad \text{(logの微分)}$$

$$= \int_\chi \frac{\partial}{\partial\theta}f(x|\theta)\,dx$$

$$= \frac{\partial}{\partial\theta}\int_\chi f(x|\theta)\,dx \quad \text{(微分可能性)}$$

$$= \frac{\partial}{\partial\theta}\cdot\int_\chi 1\cdot f(x|\theta)\,dx \quad \text{(1の期待値のDef)}$$

$$= \frac{\partial}{\partial\theta}\cdot E[1] = 0$$

$$\Rightarrow \left\{\frac{d}{d\theta}E_\theta\{W(X)\}\right\}^2 = \left\{Cov\left[W(X), \frac{\partial}{\partial\theta}\log f\right]\right\}^2$$

$$\le Var(W)\cdot Var\left(\frac{\partial}{\partial\theta}\log f\right)$$

$$Var\left(\frac{\partial}{\partial\theta}\log f\right) = E\left\{\left(\frac{\partial}{\partial\theta}\log f\right)^2\right\} - \underbrace{\left\{E\left(\frac{\partial}{\partial\theta}\log f\right)\right\}^2}_{=0\ (\Delta)}$$

$*$ $Cov\left[W(X), \frac{\partial}{\partial\theta}\log f(X|\theta)\right] = E\left[W(X)\frac{\partial}{\partial\theta}\log f(X|\theta)\right] - E[W(X)]\underbrace{E\left[\frac{\partial}{\partial\theta}\log(f(X|\theta))\right]}_{=0\ (\Delta)}$

Finally, $Var[W(X)] \ge \dfrac{\boxed{}}{\boxed{}}$

Thm 9.24 IID: $f(\mathbf{x}|\theta) = f(x_1, x_2 \dots | \theta) = \prod_{i=1}^{n} f(x|\theta) \Rightarrow$ 分母: $E_\theta\left\{\left(\frac{\partial}{\partial\theta}\log f(\mathbf{x}|\theta)\right)^2\right\} \Rightarrow n \cdot E_\theta\left\{\left(\frac{\partial}{\partial\theta}\log f(x|\theta)\right)^2\right\}$ 5

$$\underset{\Downarrow}{f(\mathbf{x}|\theta)} \qquad \underset{\Downarrow}{f(x|\theta)}$$

$$f(x_1, x_2 \dots x_n|\theta) \qquad f(x|\theta)$$

$$
\begin{aligned}
E_\theta\left\{\left(\frac{\partial}{\partial\theta}\log f(\mathbf{x}|\theta)\right)^2\right\} &= E_\theta\left\{\left(\frac{\partial}{\partial\theta}\log \prod_{i=1}^{n} f(x_i|\theta)\right)^2\right\} \\
&= E_\theta\left\{\left(\sum_{i=1}^{n} \frac{\partial}{\partial\theta}\log f(x_i|\theta)\right)^2\right\} \\
&= \sum_{i=1}^{n} E_\theta\left\{\left(\frac{\partial}{\partial\theta}\log f(x_i|\theta)\right)^2\right\} + \underline{\sum_{i\neq j} E_\theta\left\{\frac{\partial}{\partial\theta}\log f(x_i|\theta) \cdot \frac{\partial}{\partial\theta}\log f(x_j|\theta)\right\}} \quad (*)
\end{aligned}
$$

$$= \sum_{i\neq j} \underset{\substack{\| \\ 0 \ (\Delta)}}{E_\theta\left\{\frac{\partial}{\partial\theta}\log f(x_i|\theta)\right\}} \; \underset{\substack{\| \\ 0 \ (\Delta)}}{E_\theta\left\{\frac{\partial}{\partial\theta}\log f(x_j|\theta)\right\}} \qquad (x_i \perp\!\!\!\perp x_j,\ i \neq j)$$

$$= 0$$

$$= n E_\theta\left\{\left(\frac{\partial}{\partial\theta}\log f(x_i|\theta)\right)^2\right\}$$

$(*)$ $E\left\{\left(\sum_{i=1}^{n} x_i\right)^2\right\} = (x_1 + x_2 + \dots + x_n)^2 = x_1^2 + x_2^2 + \dots + x_n^2 + x_1x_2 + x_1x_3 + \cdots$

$+ x_2x_3 + x_2x_4 + \cdots$

$+ \cdots$

Thm 9.25. ⑥

$$E_\theta\left\{\frac{\partial}{\partial\theta}\log f(x|\theta)\right\}=0 \quad (\Delta)$$

$$\frac{\partial}{\partial\theta}E_\theta\left\{\frac{\partial}{\partial\theta}\log f(x|\theta)\right\}=0$$

$$=\frac{\partial}{\partial\theta}\int_{\mathcal{X}}\frac{\partial}{\partial\theta}\log f(x|\theta)\cdot f(x|\theta)\,dx$$

$$=\frac{\partial}{\partial\theta}\int_{\mathcal{X}}S(\theta,x)f(x|\theta)\,dx$$

$$=\int_{\mathcal{X}}\frac{\partial}{\partial\theta}\left[S(\theta,x)\cdot f(x|\theta)\right]dx$$

$$=\int_{\mathcal{X}}\frac{\partial}{\partial\theta}S(\theta,x)\cdot f(x|\theta)+S(\theta,x)\cdot\frac{\partial}{\partial\theta}f(x|\theta)\,dx$$

$$=\int_{\mathcal{X}}\frac{\partial}{\partial\theta}S(\theta,x)\cdot f(x|\theta)\,dx+\int_{\mathcal{X}}S(\theta,x)\cdot\frac{\partial}{\partial\theta}f(x|\theta)\,dx$$

$$=\underbrace{E_\theta\left[\frac{\partial}{\partial\theta}S(\theta,X)\right]}_{\circledast}+\int_{\mathcal{X}}\frac{\partial}{\partial\theta}\log f(x|\theta)\cdot\frac{\partial}{\partial\theta}f(x|\theta)\,dx$$

$$=\int_{\mathcal{X}}\frac{\frac{\partial}{\partial\theta}f(x|\theta)}{f(x|\theta)}\cdot\frac{\partial}{\partial\theta}f(x|\theta)\,dx$$

$$=\int_{\mathcal{X}}\frac{\left(\frac{\partial}{\partial\theta}f(x|\theta)\right)^2}{[f(x|\theta)]^2}\cdot f(x|\theta)\,dx=\int\left(\frac{\partial}{\partial\theta}\log f(x|\theta)\right)^2 f(x|\theta)\,dx=E_\theta\left\{\left(\frac{\partial}{\partial\theta}\log f(x|\theta)\right)^2\right\}$$

$\circledast$ $E_\theta\left[\frac{\partial}{\partial\theta}S(\theta,X)\right]$

$$=\int_{\mathcal{X}}\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial\theta}\log f(x|\theta)\right)f(x|\theta)\,dx$$

$$=\int_{\mathcal{X}}\frac{\partial^2}{\partial\theta^2}\log f(x|\theta)\,f(x|\theta)\,dx$$

$$=E_\theta\left[\frac{\partial^2}{\partial\theta^2}\log f(X|\theta)\right]$$

$$\Rightarrow \frac{\partial}{\partial\theta}E_\theta\left\{\frac{\partial}{\partial\theta}\log f(x|\theta)\right\}=0$$

$$=E_\theta\left[\frac{\partial^2}{\partial\theta^2}\log f(X|\theta)\right]+E_\theta\left\{\left(\frac{\partial}{\partial\theta}\log f(x|\theta)\right)^2\right\}$$

$$\Rightarrow E_\theta\left\{\left(\frac{\partial}{\partial\theta}\log f(x|\theta)\right)^2\right\}=-E_\theta\left[\frac{\partial^2}{\partial\theta^2}\log f(X|\theta)\right]$$

Eg. 9.26 ⑰

$$E_\lambda\left\{\left(\frac{\partial}{\partial\lambda}\log f(\mathbf{x}|\lambda)\right)^2\right\} \quad \leftarrow f(x_1, x_2 \cdots x_n|\theta)$$

$$= E_\lambda\left\{\left(\sum_{i=1}^{n}\frac{\partial}{\partial\lambda}\log f(x_i|\lambda)\right)^2\right\}$$

$$= n\,E_\lambda\left\{\left(\frac{\partial}{\partial\lambda}\log f(X|\lambda)\right)^2\right\}$$

$$= -n\,E_\lambda\left\{\left(\frac{\partial^2}{\partial\lambda^2}\log f(x|\lambda)\right)^2\right\} \quad (\text{Supp } 9.25)$$

$$= -n\,E_\lambda\left\{\frac{\partial^2}{\partial\lambda^2}\log\left(\frac{e^{-\lambda}\lambda^x}{x!}\right)\right\}$$

$$= -n\,E_\lambda\left\{\frac{\partial^2}{\partial\lambda^2}\left(\log e^{-\lambda} + \log\lambda^x - \log x!\right)\right\}$$

$$= -n\,E_\lambda\left\{\frac{\partial^2}{\partial\lambda^2}\left(\cancel{-\lambda} + x\log\lambda - \cancel{\log x!}\right)\right\}$$

$$= -n\,E_\lambda\left\{-\frac{x}{\lambda^2}\right\}$$

$$= -n\left(-\frac{E_\lambda[x]}{\lambda^2}\right) = n\cdot\frac{\lambda}{\lambda^2} = \frac{n}{\lambda}$$

$W(\mathbf{x})$: 不偏推定量 Thm 9.24 ⇒ 分子 $= \tau'(\lambda) = 1$

$$\mathrm{Var}_\lambda[W(\mathbf{x})] \geq \frac{1}{\left(\frac{n}{\lambda}\right)} = \frac{\lambda}{n}$$

Eg 9.29.

$$\bar{X}_n \sim N(\theta, \tfrac{1}{n})$$

$$P_\theta(|\bar{x}_n - \theta| < \varepsilon) = P_\theta(-\varepsilon+\theta < \bar{x}_n < \varepsilon+\theta)$$

$$= \int_{-\varepsilon+\theta}^{\varepsilon+\theta} \frac{1}{\sqrt{2\pi \frac{1}{n}}} \exp\left\{-\frac{1}{2}\frac{(\bar{x}_n-\theta)^2}{\frac{1}{n}}\right\} d\bar{x}_n$$

$$= \int_{\theta-\varepsilon}^{\theta+\varepsilon} \sqrt{\frac{n}{2\pi}} \exp\left\{-\frac{n}{2}(\bar{x}_n-\theta)^2\right\} d\bar{x}_n$$

$y = \bar{x}_n - \theta$

$$= \int_{-\varepsilon}^{\varepsilon} \sqrt{\frac{n}{2\pi}} \exp\left\{-\frac{n}{2}y^2\right\} dy$$

$t = \sqrt{n}\, y$

$$= \int_{-\varepsilon\sqrt{n}}^{\varepsilon\sqrt{n}} \sqrt{\frac{1}{2\pi}} \exp\left\{-\frac{1}{2}t^2\right\} dt$$

$$= P(-\varepsilon\sqrt{n} < Z < \varepsilon\sqrt{n})$$

$$= P(|Z| < \varepsilon\sqrt{n}) \xrightarrow[n\to\infty]{} 1$$

Thm 9.37. ⑨

$\tau(\theta)=\theta$ の場合.

$\ell(\theta|\mathbf{x})=\sum \log f(x_i|\theta) \to$ loglikelihood function θ の関数.

θ_0: true value of θ とする. $\hat{\theta}$: θ の MLE とする.

$$\ell'(\theta|\mathbf{x}) = \ell'(\theta_0|\mathbf{x}) + (\theta-\theta_0)\ell''(\theta_0|\mathbf{x}) + \cdots \quad \text{(Taylor Expansion)}$$

$$\underset{\substack{\| \\ 0 \\ (\hat{\theta}:\mathrm{MLE})}}{\ell'(\hat{\theta}|\mathbf{x})} = \ell'(\theta_0|\mathbf{x}) + (\hat{\theta}-\theta_0)\ell''(\theta_0|\mathbf{x}) + \cdots \quad (\hat{\theta}=\theta \text{ 代入})$$

$$\ell'(\theta_0|\mathbf{x}) = \frac{\partial}{\partial\theta}\log f(\mathbf{x}|\theta)\Big|_{\theta=\theta_0}$$

$$\sqrt{n}(\hat{\theta}-\theta_0) \approx -\frac{\ell'(\theta_0|\mathbf{x})}{\ell''(\theta_0|\mathbf{x})}\sqrt{n} = \frac{-\frac{1}{\sqrt{n}}\ell'(\theta_0|\mathbf{x})}{\frac{1}{n}\ell''(\theta_0|\mathbf{x})}$$

$I(\theta_0) = E\left(\ell'(\theta_0|\mathbf{x})^2\right) = \frac{1}{v(\theta)}$ とする.

課題

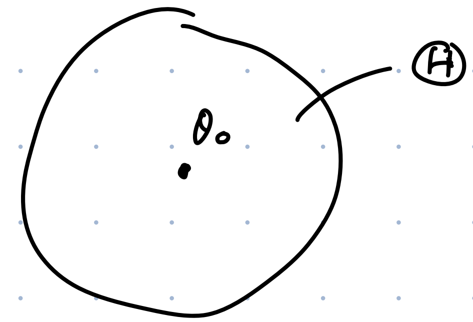
$$\begin{cases} A = \frac{1}{\sqrt{n}}\ell'(\theta_0|\mathbf{x}) = -\sqrt{n}\left(\frac{1}{n}\sum \frac{\frac{\partial}{\partial\theta}f(x_i|\theta)}{f(x_i|\theta)}\right) \xrightarrow[\mathrm{D}]{\mathrm{CLT}} N(0, I(\theta_0)) \\ B = -\frac{1}{n}\ell''(\theta_0|\mathbf{x}) = \frac{1}{n}\sum\left(\frac{\frac{\partial}{\partial\theta}f(x_i|\theta)}{f(x_i|\theta)}\right)^2 - \frac{1}{n}\sum\frac{\frac{\partial^2}{\partial\theta^2}f(x_i|\theta)}{f(x_i|\theta)} \xrightarrow[\mathrm{P}]{\mathrm{WLLN}} I(\theta_0) \end{cases}$$

$$\Rightarrow \frac{A}{B} \xrightarrow{D} N\left(0, \frac{1}{I(\theta_0)}\right)$$

10.

Eg. 10.5: $X_1 \ldots X_n \overset{iid}{\sim} N(\theta, 1)$

$H_0: \theta = \theta_0$ vs $H_1: \theta \neq \theta_0$



$\hat{\theta}_{MLE} = \underset{\theta \in \Theta}{\arg\max}\, L(\theta | \mathbf{x})$

LRT statistic $\lambda(\mathbf{x}) = \dfrac{\sup_{\theta = \theta_0} L(\theta | \mathbf{x})}{\sup_{\theta \in \Theta} L(\theta | \mathbf{x})} = \dfrac{L(\theta_0 | \mathbf{x})}{L(\hat{\theta}_{MLE} | \mathbf{x})}$

$$L(\theta | \mathbf{x}) = f(x_1 \ldots x_n | \theta) = \prod_{i=1}^{n} f(x_i | \theta) = (2\pi)^{-\frac{n}{2}} \exp\left\{ -\sum_{i=1}^{n} \frac{(x_i - \theta)^2}{2} \right\}$$

$$= \frac{(2\pi)^{-\frac{n}{2}} \exp\left\{ -\sum_i \frac{(x_i - \theta_0)^2}{2} \right\}}{(2\pi)^{-\frac{n}{2}} \exp\left\{ -\sum_i \frac{(x_i - \hat{\theta})^2}{2} \right\}} \qquad \hat{\theta} = \bar{x}$$

$$= \exp\left\{ \left(-\underbrace{\sum_{i=1}^{n} (x_i - \theta_0)^2}_{= \sum_{i=1}^{n} (x_i - \bar{x})^2 + n(\bar{x} - \theta_0)^2} + \sum_{i=1}^{n} (x_i - \bar{x})^2 \right) / 2 \right\}$$

$$= \exp\left\{ -n(\bar{x} - \theta_0)^2 / 2 \right\}$$

$$R = \{ \mathbf{x} : \lambda(\mathbf{x}) \leq c \} = \{ \mathbf{x} : \exp\{ -n(\bar{x} - \theta_0)^2 / 2 \} \leq c \} = \{ \mathbf{x} : |\bar{x} - \theta_0| \geq \sqrt{-2\log(c)/n} \}$$

Eg. 10.7

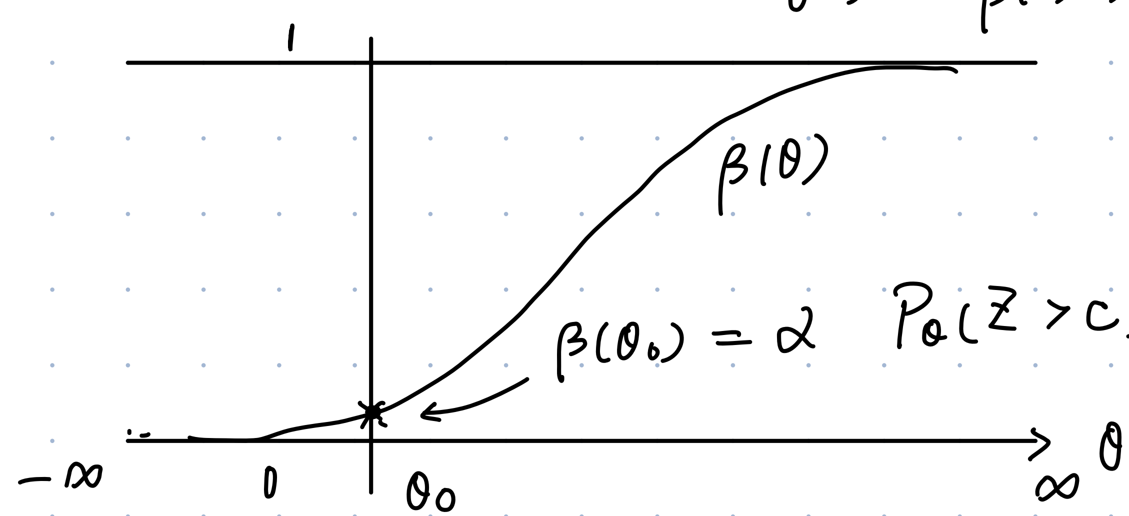
$X_1 \cdots X_n \overset{iid}{\sim} N(\theta, \sigma^2)$ ← known

$H_0: \theta \le \theta_0$ vs $H_1: \theta > \theta_0$

$R = \left\{ x : \frac{\bar{x}-\theta_0}{\sigma/\sqrt{n}} > c \right\}$ → 課題

$$\beta(\theta) = P_\theta(X \in R)$$
$$= P_\theta\left(\frac{\bar{X}-\theta_0}{\sigma/\sqrt{n}} > c\right)$$
$$= P_\theta\left(\frac{\bar{X}-\theta}{\sigma/\sqrt{n}} + \frac{\theta-\theta_0}{\sigma/\sqrt{n}} > c\right)$$
$$= P_\theta\left(Z > c - \frac{\theta-\theta_0}{\sigma/\sqrt{n}}\right)$$
$$= P_\theta\left(Z > c + \frac{\theta_0-\theta}{\sigma/\sqrt{n}}\right)$$
$$= 1 - \Phi\left(c + \frac{\theta_0-\theta}{\sigma/\sqrt{n}}\right)$$

⇒



$\theta \to \infty$ $\beta(\theta) \to 1$

$P_\theta(Z > c) = \alpha$

$\theta \to -\infty$

$\beta(\theta) \to 0$

Eg 10.9. (Eg. 10.5)

$\lambda(x) = \exp\{-n(\bar{x}-\theta_0)^2/2\}$

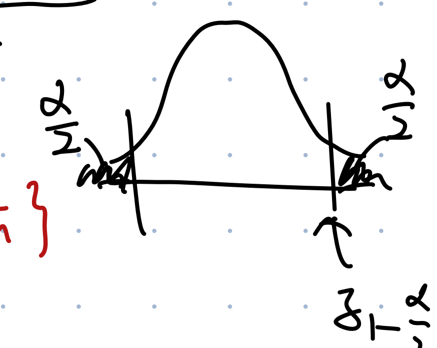
$R = \{x : \lambda(x) \le c\}$

$$\alpha = \sup_{\theta=\theta_0} \beta(\theta) = \sup_{\theta=\theta_0} P_\theta(\lambda(X) \le c)$$
$$= P_\theta\left(-\frac{n(\bar{x}-\theta_0)^2}{2} \le \log c\right)$$
$$= P_\theta\left\{ n(\bar{x}-\theta_0)^2 \ge -(2\log c) \right\}$$
$$= P_\theta\left\{ \sqrt{n}|\bar{x}-\theta_0| \ge \sqrt{-2\log c} \right\}$$
$$= P_\theta\{|Z| \ge z_{1-\frac{\alpha}{2}}\}$$
$$= P_\theta\{|\bar{x}-\theta_0| \ge z_{1-\frac{\alpha}{2}}/\sqrt{n}\}$$

$\sqrt{n}(\bar{X}-\theta_0) = \frac{\bar{X}-\theta_0}{\frac{1}{\sqrt{n}}} \sim N(0,1)$

$\| $

Z



Thm 10.15 $\qquad H_0: \theta = \theta_0 \quad vs. \quad H_1: \theta \neq \theta_0$

$$\log L(\theta|\mathbf{x}) = \ell(\theta|\mathbf{x})$$

$$= \ell(\hat\theta|\mathbf{x}) + \ell'(\hat\theta|\mathbf{x})(\theta-\hat\theta) + \ell''(\hat\theta|\mathbf{x})\frac{(\theta-\hat\theta)^2}{2!} + \cdots$$ Taylor expansion around $\hat\theta$

$$\ell(\theta_0|\mathbf{x}) = \ell(\hat\theta|\mathbf{x}) + \ell'(\hat\theta|\mathbf{x})(\theta_0-\hat\theta) + \ell''(\hat\theta|\mathbf{x})\frac{(\theta_0-\hat\theta)^2}{2!} + \cdots$$ ($\theta=\theta_0$ 代入)

$$\ell(\theta_0|\mathbf{x}) - \ell(\hat\theta|\mathbf{x}) \approx \underbrace{\ell'(\hat\theta|\mathbf{x})(\theta_0-\hat\theta)}_{=0} + \ell''(\hat\theta|\mathbf{x})\frac{(\theta_0-\hat\theta)^2}{2!}$$

$$\frac{\partial}{\partial\theta}\ell(\theta|\mathbf{x}) = 0$$

$$\Rightarrow \theta = \hat\theta \text{ (MLE)} \Leftrightarrow \ell'(\hat\theta|\mathbf{x}) = 0$$

$$\ell(\theta_0|\mathbf{x}) - \ell(\hat\theta|\mathbf{x}) \approx \frac{\ell''(\hat\theta|\mathbf{x})(\theta_0-\theta)^2}{2!}$$

Thm 9.37.

$$\sqrt{n}(\theta_0-\hat\theta) \xrightarrow{D} N\left(0, \frac{1}{I(\theta_0)}\right)$$

$$-2\log\lambda(\mathbf{x}) = -2\log\left(L(\theta_0|\mathbf{x}) - L(\hat\theta|\mathbf{x})\right)$$

$$= -2\left(\ell(\theta_0|\mathbf{x}) - 2\ell(\hat\theta|\mathbf{x})\right) \approx \underbrace{-\ell''(\hat\theta|\mathbf{x})}\,\underbrace{(\theta_0-\hat\theta)^2} \approx \{\sqrt{n}(\theta_0-\hat\theta)\}^2 I(\theta_0) = Z^2 \xrightarrow{D} (N(0,1))^2$$

$$\chi^2_1$$

$$-\frac{1}{n}\ell''(\hat\theta|\mathbf{x}) \xrightarrow{P} I(\theta_0)$$ Report 11-2(b)

Fisher information